Exploring a New Object: the Taumutation
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Abstract

We define a taumutation as an \( n \times n \) grid with exactly two different points in each row and column. A well known mathematical object is the permutation, which is defined as an ordered list of the elements 1, 2, ..., \( n \). Examples of permutations of length 4 include 1234 and 2341. By thinking of the position of an element in a permutation as an \( x \)-coordinate and setting its value to be the \( y \)-coordinate, we obtain an \( n \times n \) grid with only one point in each row and column. In a way, a taumutation is two permutations plotted on the same grid.

No one has explored our new mathematical object. In our work, we are often interested in permutations that avoid patterns. For example, permutations that avoid the pattern 132 do not have three elements from left-to-right (not necessarily consecutive), such that the first is the smallest, the second is the largest, and the third is between them. The space of permutations under pattern avoiding restrictions is well-documented; however, no one has explored our new mathematical object. In our work, we find a way to count how many taumutations exist on an \( n \times n \) grid when we avoid two permutations of length three within the grid.

Definitions

A taumutation of order \( n \) is an arrangement of points in an \( n \times n \) grid such that there are two unique points in every row and column.

Example. The following is an 8 \( \times \) 8 taumutation:

![8x8 Taumutation Example](image)

A permutation is an arrangement of the elements of a finite set. A set of \( n \) elements has \( n! \) permutations, each of length \( n \).

Example. A permutation can be plotted on an \( n \times n \) grid where element \( p_i \) is represented as a point at \((i, p_i)\). In this way, we can consider a given taumutation as two permutations plotted on top of each other. The prior example can be seen as a combination of 13542876 and 25431687.

A permutation \( p \) of length \( n \) contains a permutation \( q \) of length \( m \leq n \) as a pattern if there are \( m \) elements \( p_{i_1}, p_{i_2}, \ldots, p_{i_m} \) in \( p \) with \( i_1 < i_2 < \cdots < i_m \), such that \( p_{i_j} < p_{i_k} \) if and only if \( q_{i_j} < q_{i_k} \). Otherwise \( p \) avoids \( q \).

A taumutation \( T \) of order \( n \) contains a permutation \( p \) of length \( m \) if it can be represented as two permutations such that one contains \( p \).

The set of all permutations of length \( n \) is denoted \( S_n \). The set of permutations of length \( n \) that avoids patterns \( x \) and \( y \) is called a pattern class and is denoted \( S_n(x,y) \). Similarly, we have \( T_n \) and \( T_n(x,y) \) for taumutations.

Results

For sufficiently large \( n \), the “general case” gives a generating function for the number of taumutations of order \( n \) avoiding the given patterns. A generating function \( g(x) \) satisfies \( \sum \mu_t \mu(R^n) = g(x) \) where \( R \) is the set of patterns that are being avoided.

<table>
<thead>
<tr>
<th>Patterns</th>
<th>( n = 2 )</th>
<th>( n = 3 )</th>
<th>( n = 4 )</th>
<th>( n = 5 )</th>
<th>( n = 6 )</th>
<th>General Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>123,132</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>12</td>
<td>29</td>
<td></td>
</tr>
<tr>
<td>123,231</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>132,213</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>14</td>
<td>32</td>
<td>( \frac{x^2}{(1-x)^2} )</td>
</tr>
<tr>
<td>132,231</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>12</td>
<td>29</td>
<td>( \frac{x^2}{1-2x-x^2} )</td>
</tr>
<tr>
<td>132,312</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>12</td>
<td>29</td>
<td>( \frac{x^2}{1-2x-x^2} )</td>
</tr>
</tbody>
</table>

Diagrams and Examples

- To avoid both patterns, whenever we see a 12 pattern, the rest of the graph to the right of the 12 pattern must be at or below the level of the “1”.
- To avoid both patterns, see that no ascending graph from left to right can satisfy the necessary conditions. Thus each of these taumutations have a graph that constantly descend when moving from left to right.

Wilf Equivalence

Two taumutation classes are trivially Wilf equivalent if the graphs of patterns being avoided are rotations or reflections of one another, such as in \( T_n(213,231) \) and \( T_n(312,132) \). When we avoid two patterns of length 3, there are six taumutation classes that are not trivially Wilf equivalent to each other, so we examined one representative from each class: \( T_n(123,132), T_n(132,231), T_n(132,321), T_n(132,231), T_n(132,231), \) and \( T_n(132,312) \). When we understand the distribution of statistics on one taumutation class, then we understand the distributions on all trivially equivalent classes.

Example Proof

We will consider the taumutation class \( T_n(123,231) \). The patterns to be avoided are shown below:

![Example Patterns](image)

\[ T_n(123,231) \]

Theorem 1. \( |T_n(123,231)| = n \) for \( n \geq 3 \).

Proof. The top row of the taumutation must consist of two points in consecutive columns, and the points in each row of the underneath must fall one column to the right of the points in the above row.

Example. The number of taumutations in \( T_n(123,231) \) is \( n \). The corresponding permutations are The corresponding taumutations are:

\[ \{(1,2),(3,4),(2,3),(1,2),(1,4),(3,4),(2,3),(1,2),(1,4)\} \]

and have graphs of:

![Example Graphs](image)

References

For further reading, consult the following resources: