Exploring a New Object: the Taumutation

Deven Harris, Jake Roth, Austin Schnoor
Advisor: Dr. Lara Pudwell

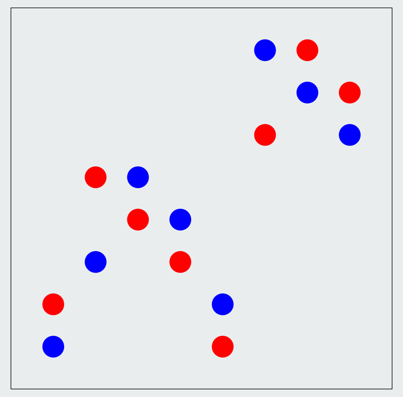
Abstract

We define a taumutation as an $n \times n$ grid with exactly two different points in each row and column. A well known mathematical object is the permutation, which is defined as an ordered list of the elements $1, 2, \ldots, n$. Examples of permutations of length 4 include 1423 and 2134. By thinking of the position of an element in a permutation as an x-coordinate and setting its value to be the y-coordinate, we obtain an $n \times n$ grid with only one point in each row and column. In a way, a taumutation is two permutations plotted on the same grid. We are often interested in permutations that avoid patterns. For example, permutations that avoid the pattern 132 do not have three elements from left-to-right (not necessarily consecutive), such that the first is the smallest, the second the largest, and the third between them. The space of permutations under pattern avoiding restrictions is well-documented; however, no one has explored our new mathematical object. In our work, we find a way to count how many taumutations exist on an $n \times n$ grid when we avoid two permutations of length three within the grid.

Definitions

- A **taumutation** of order n is an arrangement of points in an $n \times n$ grid such that there are two unique points in every row and column.

Example. The following is an 8×8 taumutation:



- A **permutation** is an arrangement of the elements of a (finite) set. A set of n elements has $n!$ permutations, each of length n.

Example. A permutation can be plotted on an $n \times n$ grid where element p_i is represented as a point at (i, p_i). In this way, we can consider a given taumutation as two permutations plotted on top of each other. The prior example can be seen as a combination of 13542876 and 25431687.

- A permutation p of length n contains a permutation q of length $m \leq n$ as a **pattern** if there are m elements $p_{i_1}, p_{i_2}, \ldots, p_{i_m}$ in p with $i_1 < i_2 < \cdots < i_m$ such that $p_{i_a} < p_{i_b}$ if and only if $q_a < q_b$. Otherwise p avoids q.
- A taumutation τ of order n contains a permutation p of length m if it can be represented as two permutations such that one contains p.
- The set of all permutations of length n is denoted S_n. The set of permutations of length n that avoids patterns x and y is called a **pattern class** and is denoted $S_n(x, y)$. Similarly, we have T_n and $T_n(x, y)$ for taumutations.

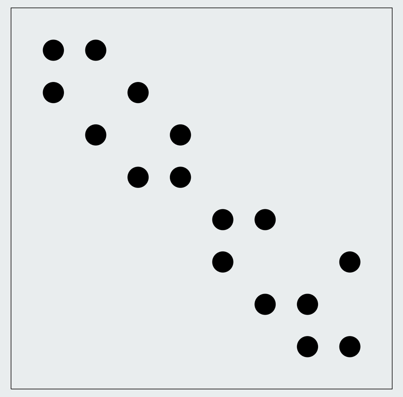
Results

For sufficiently large n, the "general case" gives a generating function for the number of taumuations of order n avoiding the given patterns. A generating function $g(x)$ satisfies $\sum_{n=0}^{\infty} t_n(R)x^n = g(x)$ where R is the set of patterns that are being avoided.

Patterns	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	General Case
123,132	1	2	5	12	29	$\frac{x^2}{1-2x-x^2}$
123,231	1	3	4	5	6	$\frac{x^2(1+x-x^2)}{(1-x)^2}$
132,213	1	3	6	14	32	$\frac{-(x^3-x^2+1)x^2}{x^5-x^4+x^3-2x^2+3x-1}$
132,231	1	2	5	12	29	$\frac{x^2}{1-2x-x^2}$
132,312	1	2	5	12	29	$\frac{x^2}{1-2x-x^2}$

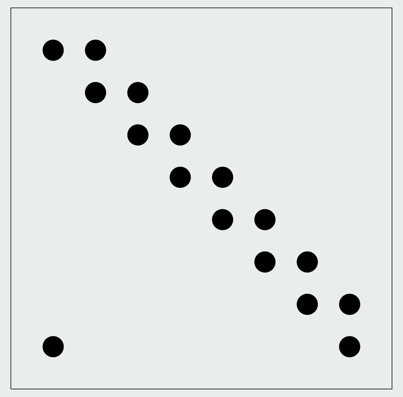
Diagrams and Examples

$T_8(123, 132)$



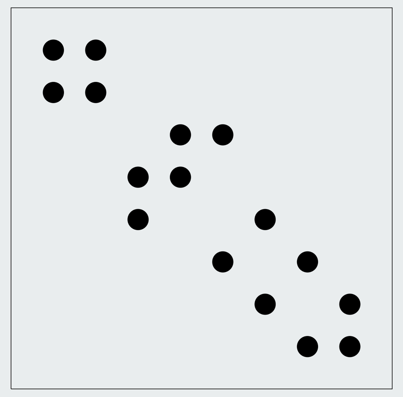
- To avoid both patterns, whenever we see a 12 pattern, the rest of the graph to the right of the 12 pattern must be at or below the level of the "1".

$T_8(123, 231)$



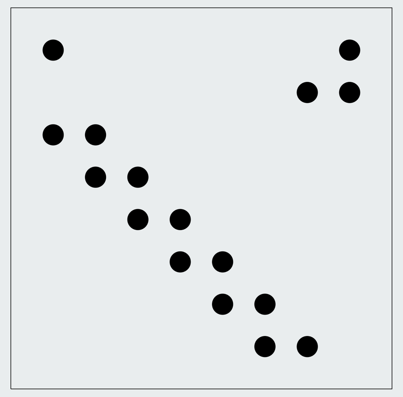
- To avoid both patterns, we see that no ascending graph from left to right can satisfy the necessary conditions. Thus each of these taumutations have graphs that constantly descend when moving from left to right.

$T_8(132, 213)$



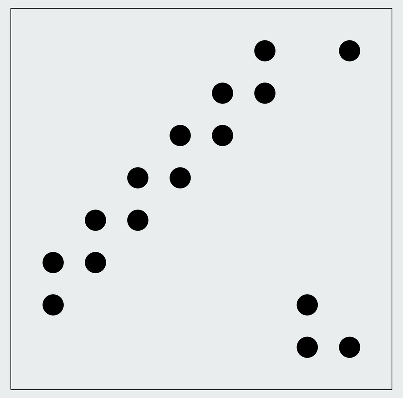
- To avoid both patterns, any consecutive 12 pattern must consist of consecutive values. If not, then we are guaranteed to have one of the two patterns. Hence, we see that each taumutation must have a general downward trend.

$T_8(132, 231)$



- Taumutations that avoid 132 and 231 must always have their largest values at the very first or very last columns of the graph. Notice that these taumutations can be obtained by rotating the graphs taumutations avoiding 132 and 312 by 90°.

$T_8(132, 312)$



- To avoid both patterns, points, when following from left to right, points may not rise or fall unless the third point in the ascension or descension does not fall between the first two points in terms of height. One thing all of these taumutations have in common is a far-right column with points at height 1 and height n.

$T_8(123, 321)$

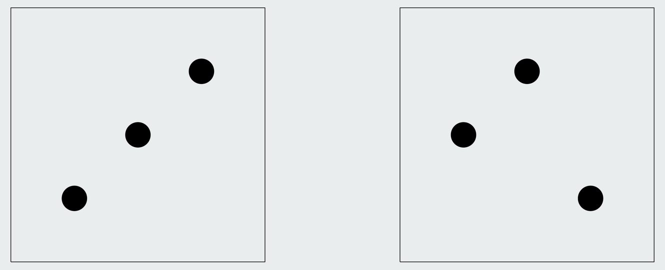
- This diagram is blank because permutations (and hence taumutations) with $n \geq 5$ must contain at least one of these two patterns. Therefore, $T_8(123, 321)$ is empty.

Wilf Equivalence

Two taumutation classes are **trivially Wilf equivalent** if the graphs of patterns being avoided are rotations or reflections of one another, such as in $T_n(213, 231)$ and $T_n(312, 132)$. When we avoid two patterns of length 3, there are six taumutation classes that are not trivially Wilf equivalent to each other, so we examined one representative from each class: $T_n(123, 132)$, $T_n(123, 231)$, $T_n(123, 321)$, $T_n(132, 213)$, $T_n(132, 231)$, and $T_n(132, 312)$. When we understand the distribution of statistics on one taumutation class, then we understand the distributions on all trivially equivalent classes.

Example Proof

We will consider the taumutation class $T_n(123, 231)$. The patterns to be avoided are shown below. .



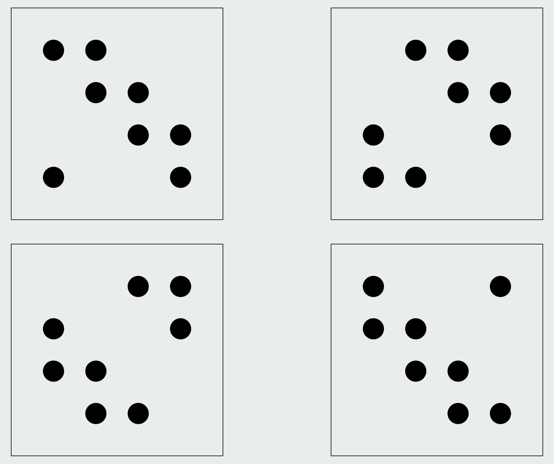
Theorem 1. $|T_n(123, 231)| = n$ for $n \geq 3$.

Proof. The top row of the taumutation must consist of two points in consecutive columns, and the points in each row of the underneath must fall one column to the right of the points in the above row. □

Example. The number of taumutations in $T_4(123, 231)$ is $n = 4$. The corresponding permutations are The corresponding taumutations are

$$[\{1,4\},\{3,4\},\{2,3\},\{1,2\}], [\{1,2\},\{1,4\},\{3,4\},\{2,3\}],$$
$$[\{2,3\},\{1,2\},\{1,4\},\{3,4\}], [\{3,4\},\{2,3\},\{1,2\},\{1,4\}],$$

and have graphs of:



References

For further reading, consult the following resources:

- R. Simion and F. Schmidt, Restricted Permutations, *European J. Combin.* **6** (1985), 383–406.
- N. J. A. Sloane, editor, The On-Line Encyclopedia of Integer Sequences, published electronically at `https://oeis.org`.