Non-Decreasing Sequences
Alexander Bruno and Amy Klass
Advisor: Dr. Jon Beagley

Abstract
Non-decreasing sequences are a generalization of binary covering arrays, which has made research on non-decreasing sequences important in both math and computer science. The goal of this research is to find properties of these non-decreasing sequences as the variables d, s, and t change. The goal is also to explore methods for creating a maximum length non-decreasing sequence for a given strength and size set. Through our research, we discovered and proved basic properties of these non-decreasing sequences. In addition to this, we can describe a method we used while trying to find the maximum length of a sequence.

Definitions and Notation
• Let $S$ be a set of elements.
• The strength of a non-decreasing sequence is the amount of subsets whose union we consider, and is represented using $d$.
• A non-decreasing sequence of strength $d$ is a sequence of non-empty subsets, $\{S_1, S_2, \ldots, S_t\}$, where the union of any $d$ previous subsets does not contain any subsequent subset.
• The number of subsets in a non-decreasing sequence is called the length, $t$.
• $NDS(d, s, t)$ is the set of non-decreasing sequences with strength $d$, $s$ elements and length $t$.
• $NDST(d, s)$ is the maximum $t$ such that $NDS(d, s, t)$ is non-empty.
• Let $r_j$ be the number of elements in the subset $S_j$.

Binary Arrays
• Represent a non-decreasing sequence using an $s \times t$ binary array.
• Rows represent elements of $S$.
• Columns represent subsets of non-decreasing sequence.

<table>
<thead>
<tr>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_3$</th>
<th>$S_4$</th>
<th>$S_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Found Upper Bound
$NDS(d, s, t)$ is the set of non-decreasing sequences.

Bounds
• Gives range for $NDST(d, s)$.
• Lower bound is the length of sequence constructed for a given $d$, $s$.
• Upper bound initially $2^s - 1$, number of nonempty subsets possible for any set $S$ with $s$ elements.
• Upper bound decreased using Theorems 7, 8, and 9.

Basic Results
Theorem 1—Permuting rows in a binary array gives another $NDS(d, s, t)$.
Theorem 2—If the union of any $d$ subsets contain all elements in $S$, no subsets can be added to the sequence.
Theorem 3—Every subset in $NDS(d, s, t)$ must be distinct for $d \geq 1$.
Theorem 4—$NDS(d, s, t) \subseteq NDS(d, s + 1, t)$

Corollary 5—$NDST(d, s) \leq kNDST(d, s)$, where $k \in \mathbb{Z}$.

Standard Sequence
Theorem 6—There exists an $NDS(d, s, t)$ where the first $s$ subsets are of size 1. We call this a standard non-decreasing sequence.

$$\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 \\
3 & 0 & 0 & 1 & 0 & 0 \\
4 & 0 & 0 & 0 & 1 & 0 \\
5 & 0 & 0 & 0 & 0 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}$$

<table>
<thead>
<tr>
<th>Lower Bound</th>
<th>Bound Upper Bound</th>
<th>$2^s - 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>15</td>
</tr>
<tr>
<td>5</td>
<td>7</td>
<td>31</td>
</tr>
<tr>
<td>6</td>
<td>11</td>
<td>63</td>
</tr>
<tr>
<td>7</td>
<td>15</td>
<td>127</td>
</tr>
</tbody>
</table>

Table 1: Bounds for $d = 2$

Future Work
• Find exact formula for $NDST(d, s)$
• Find different computational methods
• Find relation to binary covering arrays
• Effect of permuting columns
• Find bounds for larger $d$ and $s$ values

References

Acknowledgements
MSSEED Program sponsored by NSF (Grant No. 1068346)
Valparaiso University Mathematics and Statistics Department and Professor Jon Beagley