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Noise-Induced Stabilization of Perturbed Hamiltonian Systems

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Abstract. Noise-induced stabilization is the phenomenon in which the addition of randomness to an unstable system of ordinary differential equations results in a stable system of stochastic differential equations. With stability defined as global stochastic boundedness, Hamiltonian systems can never be stabilized by the addition of noise that is constant in space. In this article, we investigate how to deterministically perturb a class of unstable Hamiltonian systems in such a way that the qualitative behavior is preserved, but that enables the systems to exhibit noise-induced stabilization.

1. INTRODUCTION. A Hamiltonian system on $\mathbb{R}^2$ is a system of ordinary differential equations (ODEs) of the form

$$\begin{cases} \dfrac{dx(t)}{dt} = \dfrac{\partial H}{\partial y}(x(t), y(t)) \\ \dfrac{dy(t)}{dt} = -\dfrac{\partial H}{\partial x}(x(t), y(t)), \end{cases}$$

where $H(x, y) \in C^\infty(\mathbb{R}^2)$ is called the Hamiltonian function. A key property of Hamiltonian systems is that

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial y}\frac{dy}{dt} = \frac{\partial H}{\partial x}\frac{\partial H}{\partial y} - \frac{\partial H}{\partial y}\frac{\partial H}{\partial x} = 0, \tag{1}$$

and hence the Hamiltonian function is constant along each solution curve. In applications, H typically represents the energy of the system and (1) indicates that H is a conserved quantity.

Random perturbations $\xi_x(t)$ and $\xi_y(t)$ can be added to a Hamiltonian system to form a system of equations of the form

$$\begin{cases} \dfrac{dx(t)}{dt} = \dfrac{\partial H}{\partial y}(x(t), y(t)) + \xi_x(t) \\ \dfrac{dy(t)}{dt} = -\dfrac{\partial H}{\partial x}(x(t), y(t)) + \xi_y(t). \end{cases}$$

A common approach is to model the random perturbations as Gaussian white noise, setting $\xi_x(t) = \epsilon_x \frac{dB_t^x}{dt}$ and $\xi_y(t) = \epsilon_y \frac{dB_t^y}{dt}$, where B_t^x and B_t^y are independent Brownian motions and ϵ_x and ϵ_y control the strength of the noise in the x- and y-directions, respectively. Brownian motion can be defined as the unique stochastic process with continuous paths whose increments are stationary, independent, and have mean zero. These defining characteristics of Brownian motion make its derivative, the Gaussian white noise process, a desirable model for random fluctuations, but the derivative of

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Brownian motion exists only in the sense of distributions [6, pp. 21–29]. Thus, it is typical to write the resulting system of stochastic differential equations (SDEs) in the form

$$\begin{cases} dX_t = \dfrac{\partial H}{\partial y}(X_t, Y_t)dt + \epsilon_x dB_t^x \\ dY_t = -\dfrac{\partial H}{\partial x}(X_t, Y_t)dt + \epsilon_y dB_t^y. \end{cases} \tag{2}$$

The system of SDEs is in fact shorthand for a system of integral equations, where integration with respect to Brownian motion can be defined rigorously with the Ito integral. The Ito integral is constructed analogously to the Riemann–Stieltjes integral, but extended to the case of integrating with respect to Brownian motion, which does not have bounded variation.

For any initial condition $(X_0, Y_0) \in \mathbb{R}^2$, there exists a unique solution to (2) until the possible time of explosion (i.e., when the solution tends to infinity in finite time) since H is smooth, and hence its partial derivatives are locally Lipschitz continuous [6, pp. 68–69]. While there are many different notions of stability, we are interested here in a notion of global stability of systems, which is given in the following definition.

Definition (Stable). A system is *stable* if, for all $(X_0, Y_0) \in \mathbb{R}^2$ and for all $\delta > 0$, there exists a bound M such that $P(|(X_t, Y_t)| \leq M) > 1 - \delta$ for all $t \geq 0$.

This definition of stable is often referred to as *stochastic boundedness* or *bounded in probability*, and reduces to the standard definition of bounded when (X_t, Y_t) is deterministic (i.e., when $\epsilon_x = \epsilon_y = 0$). We say that a system of ODEs exhibits *noise-induced stabilization* if the ODE system is unstable but, after the addition of noise, the corresponding SDE system is stable. This phenomenon of noise-induced stabilization is quite intriguing since one's first intuition is often that noise would only serve to further destabilize a system, rather than have a stabilizing effect.

A classic example of noise-induced stabilization in one dimension involves the simple ODE

$$\frac{dx(t)}{dt} = rx(t), \tag{3}$$

whose solution $x(t) = x(0)e^{rt}$ is stable when $r \leq 0$, but unstable when $r > 0$. If we randomly perturb the ODE with constant white noise to form the SDE

$$dX_t = rX_t dt + \epsilon dB_t,$$

then the solution is called the *Ornstein–Uhlenbeck process* and has the form

$$X_t = X_0 e^{rt} + \epsilon e^{rt} \int_0^t e^{-rs} dB_s.$$

When $\epsilon \neq 0$, the stability classification of the Ornstein–Uhlenbeck process is almost identical to that of the original ODE in that the process is still stable when $r < 0$ and unstable when $r > 0$; however, the process is now also unstable when $r = 0$. Thus, the addition of constant white noise is not sufficient to stabilize the ODE given in equation (3). If we instead randomly perturb the ODE with noise whose magnitude depends upon space to form the SDE

$$dX_t = rX_t dt + \epsilon X_t dB_t,$$

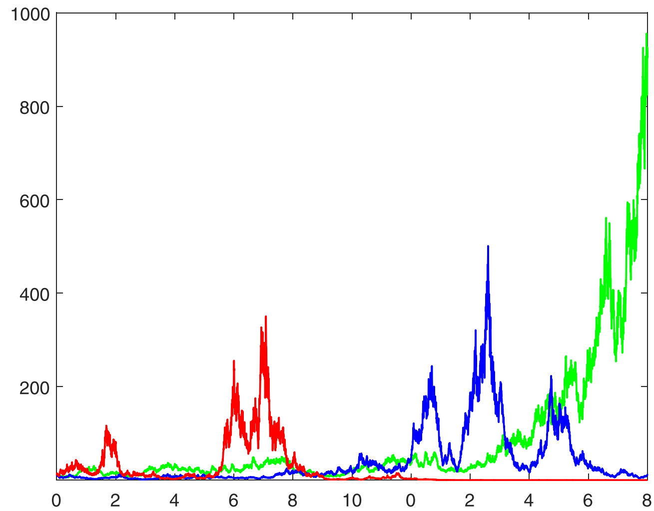


Figure 1. Simulation of geometric Brownian motion with $r = 1$ and $\epsilon = 1$ (green), $\sqrt{2}$ (blue), 2 (red).

then the solution is called *geometric Brownian motion*. The explicit solution for geometric Brownian motion is

$$X_t = X_0 e^{(r - \frac{\epsilon^2}{2})t + \epsilon B_t}$$

and its three distinct possible behaviors are depicted in Figure 1. When $\epsilon^2 < 2r$, geometric Brownian motion converges to plus or minus infinity with probability one. When $\epsilon^2 = 2r$, the solution fluctuates between arbitrary large and arbitrary small values (or remains constant if $\epsilon = r = 0$). Finally, when $\epsilon^2 > 2r$, the solution converges to zero with probability one. Thus, geometric Brownian motion exhibits noise-induced stabilization in the case where $\epsilon^2 > 2r$ and $r > 0$ since the corresponding deterministic system is unstable while the stochastic system is stable.

In [**8**], Scheutzow proved that a one-dimensional ODE that explodes in finite time can never be stabilized by noise that is constant in space. Yet in two dimensions this type of stabilization is possible, which has been demonstrated in several works [**2**,**4**,**7**]. The examples of noise-induced stabilization in the plane typically involve systems where the regions in which the deterministic dynamics point outward toward infinity are isolated, and the addition of white noise allows the solution to escape the unstable regions into regions where the deterministic dynamics flow inward toward the origin. While it is possible to have noise-induced stabilization of two-dimensional systems with additive white noise, an unstable two-dimensional *Hamiltonian* system can never be stabilized by noise that is constant in space. Due to the Hamiltonian structure, Lebesgue measure will always be invariant, and hence the solution visits all points in the plane with equal probability rather than remaining bounded with high probability.

In this article, we consider a class of unstable Hamiltonian systems and investigate how to deterministically perturb them in such a way that the qualitative behavior of the deterministic systems is preserved, but that enables the systems to exhibit noise-induced stabilization. Section 2 describes the behavior and instability of the deterministic Hamiltonian and perturbed Hamiltonian systems under consideration. In Section 3, we show that the deterministically perturbed Hamiltonian systems do indeed exhibit noise-induced stabilization by proving that the corresponding stochastic systems are stable.

2. DETERMINISTIC SETTING. We consider the class of Hamiltonian systems with Hamiltonian function of the form $H(x, y) = h(x^m y^n)$, where $m, n \geq 2$ are integers and $h \in C^\infty(\mathbb{R})$. We also assume that h' is bounded away from zero, i.e., there exists $a > 0$ such that $|h'(t)| \geq a$ for all t. For any Hamiltonian function of this form, the corresponding system of ODEs is given by

$$\begin{cases} \dfrac{dx}{dt} = h'(x^m y^n) n x^m y^{n-1} \\ \dfrac{dy}{dt} = -h'(x^m y^n) m x^{m-1} y^n \end{cases} \tag{4}$$

with initial condition $(x(0), y(0)) = (x_0, y_0)$. We consider this particular class of Hamiltonian systems due to its intriguing mathematical nature, which belies the simple structure of the Hamiltonian function; namely, every Hamiltonian system in the class is unstable, but the qualitative behavior suggests that a small perturbation may result in stabilization. The precise behavior of this class of Hamiltonian systems is described in more detail below.

With $m, n \geq 2$, both axes consist of a continuum of equilibrium points. However, if $m = n = 1$, the qualitative behavior is different since only the origin is an equilibrium; more generally, at most one of the axes consists of a continuum of equilibria if $m = 1$ or $n = 1$, and hence we exclude this case. Due to the fact that the Hamiltonian function is constant along each solution curve and the fact that our restrictions on h imply that h is invertible, we know that $x(t)^m y(t)^n = x_0^m y_0^n$ for all t, and thus, for any initial condition off the axes, $y(t) = y_0 \left| \frac{x_0}{x(t)} \right|^{m/n}$.

The four possible phase portraits for the Hamiltonian systems with $m = n$ are depicted in Figure 2. While the shape of the solution curves is the same in all cases, the direction of the arrows depends upon the sign of h' and whether $m = n$ is even or odd. When $m \neq n$, there is no longer symmetry in the shape of the solution curves near the x- and y-axes, and there are more possible combinations for the directions of the arrows due to different combinations of m and n being even or odd.

If $m = n$, the explicit solution to (4) is given by

$$\begin{cases} x(t) = x_0 \exp[h'((x_0 y_0)^n) n (x_0 y_0)^{n-1} t] \\ y(t) = y_0 \exp[-h'((x_0 y_0)^n) n (x_0 y_0)^{n-1} t]. \end{cases}$$

From the explicit solution, we can observe that for all initial conditions off the axes, either $\lim_{t\to\infty} |x(t)| = \infty$ or $\lim_{t\to\infty} |y(t)| = \infty$. Thus, the deterministic Hamiltonian system is unstable. The only effect of n is determining the precise exponential rate at which the solution approaches one of the axes.

If $m \neq n$, the explicit solution to (4) is not as obvious, but can be found by considering $z(t) = x(t)^{m-1} y(t)^{n-1}$, which satisfies the ODE

$$\frac{dz}{dt} = h'(x^m y^n)(m-n) x^{2m-2} y^{2n-2} = h'(x_0^m y_0^n)(m-n) z^2. \tag{5}$$

Solving the ODE in (5) yields

$$z(t) = \frac{x_0^{m-1} y_0^{n-1}}{1 - h'(x_0^m y_0^n)(m-n) x_0^{m-1} y_0^{n-1} t}. \tag{6}$$

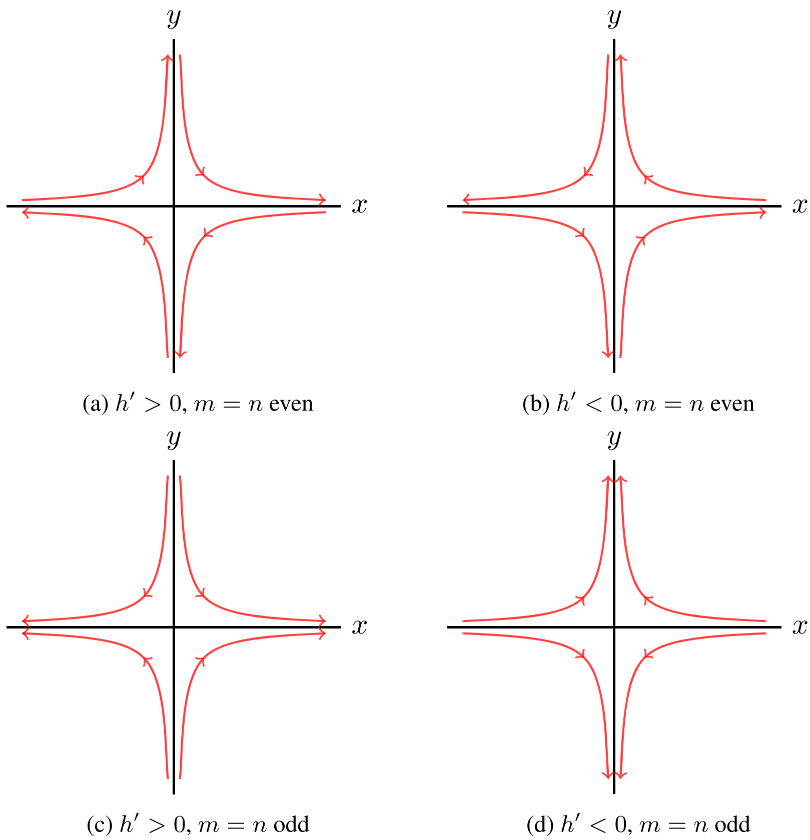


Figure 2. Possible phase portraits for the Hamiltonian systems with $m = n$.

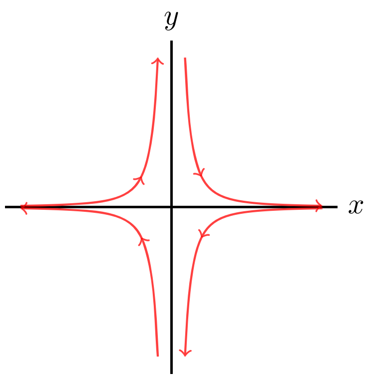
When $m = n$, $z(t)$ is a constant, but when $m \neq n$ we can plug the expression for $z(t)$ given in (6) into the system of ODEs given in (4) to find

$$\begin{cases} x(t) = x_0(1 - h'(x_0^m y_0^n)(m-n)x_0^{m-1}y_0^{n-1}t)^{\frac{-n}{m-n}} \\ y(t) = y_0(1 - h'(x_0^m y_0^n)(m-n)x_0^{m-1}y_0^{n-1}t)^{\frac{m}{m-n}}. \end{cases}$$

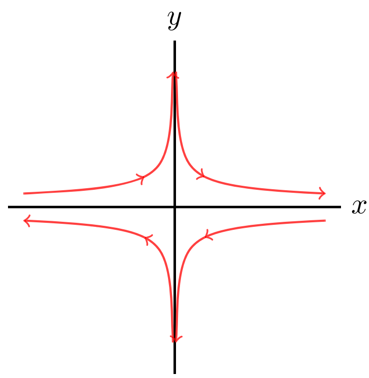
As in the case when $m = n$, we still observe that for all initial conditions off the axes, either $\lim_{t\to\infty} |x(t)| = \infty$ or $\lim_{t\to\infty} |y(t)| = \infty$. However, when $m \neq n$ the nature of the instability is stronger since there exist initial conditions for which the solution actually blows up in finite time. In particular, the sign of

$$h'(x_0^m y_0^n)(m-n)x_0^{m-1}y_0^{n-1}$$

determines the nature of the instability, with a positive sign corresponding to a solution that explodes in finite time and a negative sign corresponding to a solution that simply wanders off to infinity. For example, if $h' > 0$ and $m > n$ with both m and n even, solutions starting in the first or third quadrants have $|x(t)|$ blowing up in finite time, while solutions starting in the second or fourth quadrants have $|y(t)|$ approaching infinity at a rate of order $t^{\frac{m}{m-n}}$. If instead $m < n$, $|y(t)|$ blows up in finite time in the second and fourth quadrants, while $|x(t)|$ approaches infinity at a rate of order $t^{\frac{n}{n-m}}$ in the first and third quadrants. These two cases are depicted in Figure 3, where the quadrants that exhibit finite-time explosion correspond to the quadrants in which the solution approaches the axis more quickly.



(a) $m > n$



(b) $m < n$

Figure 3. Phase portraits for the Hamiltonian systems with $h' > 0$, $m \neq n$ even.

Although the system given by (4) is unstable for any values of m and n, in the case where m and n are even, as depicted in Figure 2(a) and (b) and in Figure 3, the system possesses characteristics which may lead one to guess that it will exhibit noise-induced stabilization. For the deterministic process, solutions remain in the quadrant in which they start for all time, approaching infinity along one of the axes. In contrast, one might guess that the addition of white noise will enable the stochastic process to cross the axes and form a quasi-periodic orbit where the process continually traverses all four quadrants in a clockwise motion if $h' > 0$ or a counterclockwise motion if $h' < 0$.

However, as mentioned in Section 1, Lebesgue measure is always invariant for a Hamiltonian system with constant noise, and hence stabilization by constant noise is not mathematically possible. Figure 4 displays a simulation of the Hamiltonian system with constant noise ($\epsilon_x = \epsilon_y = 1$) added, in the case where $m = n$ are even and the initial condition is in the first quadrant. From the figure, one can observe that the noise does indeed allow the process to traverse all four quadrants, but the system is still unstable because the process still tends toward infinity along the axes. In fact, while the figure displays only the range from $(-200, 200)$, the simulation eventually results in numerical overflow when run for a sufficient amount of time.

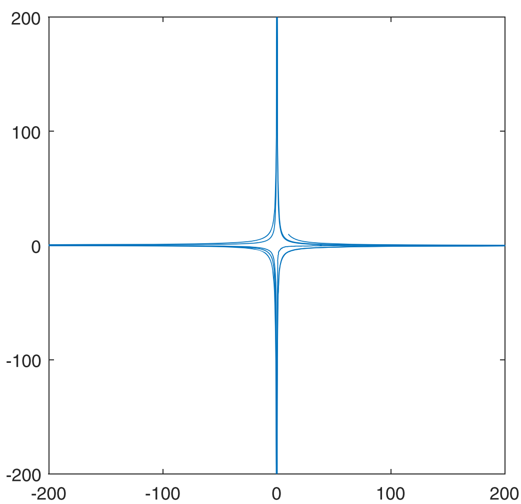


Figure 4. Simulation of the Hamiltonian system with constant noise added.

Since the Hamiltonian system cannot be stabilized by constant noise, we seek to deterministically perturb the system to break the Hamiltonian structure and allow for noise-induced stabilization to occur. In order to choose the precise form for the deterministic perturbation, we again consider the function $z(t) = x(t)^{m-1}y(t)^{n-1}$, which was crucial for analyzing the mathematical behavior of the original Hamiltonian system. We know from the expression derived in (6) that $z(t)$ either always remains constant (when $m = n$) or explodes in finite time for certain initial conditions (when $m \neq n$). However, in order for $(x(t), y(t))$ to cross the axes, $z(t)$ must be able to reach zero. Hence, our deterministic perturbation is constructed in order to give $z(t)$ an extra push toward zero, but without altering the set of equilibrium points of the original Hamiltonian system. This desired effect is achieved with the following perturbed system:

$$\begin{cases} \dfrac{dx}{dt} = h'(x^m y^n)nx^m y^{n-1} - (h'(x^m y^n))^2 nx^{2m-1}y^{2n-2} \\ \dfrac{dy}{dt} = -h'(x^m y^n)mx^{m-1}y^n - (h'(x^m y^n))^2 mx^{2m-2}y^{2n-1}. \end{cases} \tag{7}$$

With this specific perturbation, $z(t)$ now satisfies the following ODE:

$$\frac{dz}{dt} = h'(x^m y^n)(m-n)z^2 - (h'(x^m y^n))^2(n(m-1) + m(n-1))z^3. \tag{8}$$

From equation (8), we can observe that the deterministic perturbation will preserve the instability and essential limiting behavior of the original Hamiltonian system because the new term involving z^3 is not strong enough to enable $z(t)$ to reach zero in finite time. In the next section, we prove that the deterministic perturbation does indeed allow the system to be stabilized by noise that is constant in space.

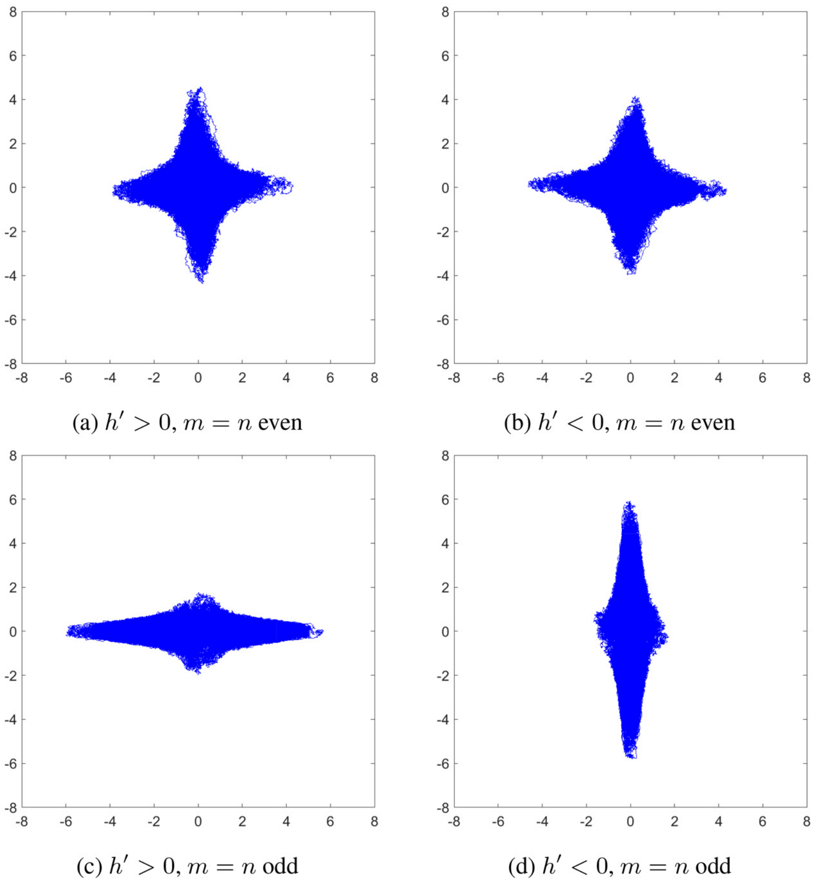
3. STOCHASTIC SETTING. We now consider adding white noise to the deterministically perturbed system given in (7) in order to form the system of SDEs that appears in the theorem below.

Theorem 1. *The system of SDEs*

$$\begin{cases} dX_t = [h'(X_t^m Y_t^n)X_t^{m-1}Y_t^{n-1} - (h'(X_t^m Y_t^n)X_t^{m-1}Y_t^{n-1})^2]nX_t dt + \epsilon_x dB_t^x \\ dY_t = [-h'(X_t^m Y_t^n)X_t^{m-1}Y_t^{n-1} - (h'(X_t^m Y_t^n)X_t^{m-1}Y_t^{n-1})^2]mY_t dt + \epsilon_y dB_t^y \end{cases}$$

exhibits noise-induced stabilization; i.e., the system is unstable when $\epsilon_x = \epsilon_y = 0$, but is stable whenever $\epsilon_x \neq 0$ and $\epsilon_y \neq 0$.

We have already shown the instability of the deterministic system when $\epsilon_x = \epsilon_y = 0$ in Section 2, so in order to prove Theorem 1, it remains to show that the stochastic system with $\epsilon_x \neq 0$ and $\epsilon_y \neq 0$ is stable. Figure 5 shows simulations of the stochastic system with $\epsilon_x = \epsilon_y = 1$ and $m = n$. The simulations provide visual evidence of stabilization since, rather than converging to infinity along one of the axes, the solutions remain bounded. The reason for the different shapes in Figure 5 is apparent by comparison with the corresponding phase portraits of the original Hamiltonian systems depicted in Figure 2. When m and n are even, there is symmetry along the x- and y-axes since the deterministic drift points either clockwise (when $h' > 0$) or counterclockwise (when $h' < 0$). However, when m and n are odd, the deterministic drift points either outward in the x-direction (when $h' > 0$) or outward in the y-direction (when $h' < 0$). Figure 6 displays a simulation of the stochastic system with $\epsilon_x = \epsilon_y = .001$, $h' > 0$, and $m = n$



(a) $h' > 0, m = n$ even

(b) $h' < 0, m = n$ even

(c) $h' > 0, m = n$ odd

(d) $h' < 0, m = n$ odd

Figure 5. Simulations of the stochastic system with $\epsilon_x = \epsilon_y = 1, m = n$.

even, zoomed in about the origin, where the clockwise quasi-periodic motion starting from initial condition (.1, .1) is more easily visible.

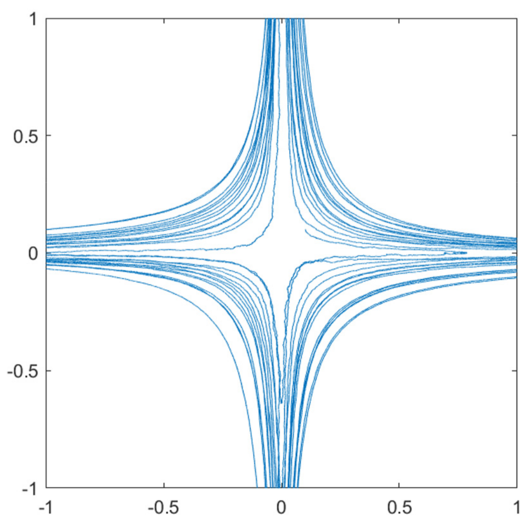


Figure 6. Simulation of the stochastic system with $\epsilon_x = \epsilon_y = .001, h' > 0, m = n$ even.

To rigorously prove stability of the stochastic system, we use the well-known result that the existence of a global Lyapunov function satisfying the definition below implies that the system is stable [5].

Definition (Lyapunov function). A function $V(x, y)$ is a *local Lyapunov function* on $\mathcal{R} \subset \mathbb{R}^2$ if it satisfies

1. $V \in C^\infty(\mathcal{R})$,
2. $\lim_{r\to\infty}\left[\inf_{(x,y)\in(\mathcal{R}\cap B_r^c)} V(x, y)\right] = \infty$,
3. $\lim_{r\to\infty}\left[\sup_{(x,y)\in(\mathcal{R}\cap B_r^c)} (\mathcal{L}V)(x, y)\right] = -\infty$,

where B_r^c is the complement of the ball of radius r about the origin and $\mathcal{L}$ is the generator corresponding to the system. If $\mathcal{R} = \mathbb{R}^2$, we say that $V(x, y)$ is a *global Lyapunov function*.

Note that the generator $\mathcal{L}$ of a stochastic process (X_t, Y_t) is defined by

$$(\mathcal{L}V)(x, y) = \lim_{t\to 0} \frac{E^{(x,y)}[V(X_t, Y_t)] - V(x, y)}{t},$$

where $E^{(x,y)}[\cdot]$ denotes the expected value starting from point (x, y). Hence, the generator describes the expected movement of the stochastic process in an infinitesimal time interval. When applied to functions V that are twice continuously differentiable, the generator can be represented as a second-order partial differential operator [6, pp. 121–124]. In particular, the generator corresponding to the system of SDEs considered in Theorem 1 is given by

$$\mathcal{L} = [(h'(x^m y^n)x^{m-1}y^{n-1}) - (h'(x^m y^n)x^{m-1}y^{n-1})^2]nx\frac{\partial}{\partial x} + \frac{\epsilon_x^2}{2}\frac{\partial^2}{\partial x^2}$$
$$+ [(h'(x^m y^n)x^{m-1}y^{n-1}) - (h'(x^m y^n)x^{m-1}y^{n-1})^2]my\frac{\partial}{\partial y} + \frac{\epsilon_y^2}{2}\frac{\partial^2}{\partial y^2}.$$

Dynkin's formula states that

$$E^{(x,y)}[V(X_t, Y_t)] = V(x, y) + E^{(x,y)}\left[\int_0^t (\mathcal{L}V)(X_s, Y_s)ds\right],$$

and this formula gives intuition for why the conditions in the definition of a global Lypaunov function imply stability. The condition that $(\mathcal{L}V)(X_s, Y_s)$ tends toward negative infinity as the magnitude of (X_s, Y_s) approaches infinity ensures that $E^{(x,y)}[V(X_t, Y_t)]$ is bounded above by $V(x, y)$ for large initial conditions (x, y). Yet, because of the condition that $V(X_t, Y_t)$ approaches positive infinity as the magnitude of (X_t, Y_t) approaches infinity, the magnitude of (X_t, Y_t) itself must be stochastically bounded, which is our definition of stable in this context.

Generally, showing the existence of a Lyapunov function can be quite ad hoc and tedious. However, we apply the systematic method developed in [2] in order to construct local Lyapunov functions on various regions of the plane and then patch them together to form one smooth, global Lyapunov function.

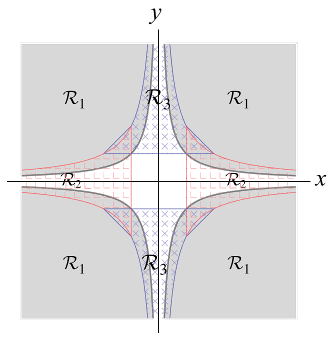


Figure 7. Decomposition of plane into priming and diffusive regions.

We begin by decomposing the plane into the following regions:

$$\mathcal{R}_1 = \{(x, y) : |x|^{m-1}|y|^{n-1} \geq c\},$$
$$\mathcal{R}_2 = \{(x, y) : |x|^{m-1}|y|^{n-1} \leq 2c, |x| \geq 1\},$$
$$\mathcal{R}_3 = \{(x, y) : |x|^{m-1}|y|^{n-1} \leq 2c, |y| \geq 1\},$$

where $c > 0$. The precise value of the constant c will be specified later to facilitate local Lyapunov function constructions and patching. The three regions, $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$, cover the entire plane, minus some ball about the origin, and are depicted in Figure 7 in the case of $m = n$. $\mathcal{R}_1$ is the "priming region" where a natural Lyapunov function exists, namely the norm to some power, which we prove in Lemma 2. $\mathcal{R}_2$ and $\mathcal{R}_3$ are "diffusive regions" where the deterministic dynamics are unstable and noise is essential to the existence of local Lyapunov functions, which we prove in Lemma 3. In Lemma 4, we prove that the local Lyapunov functions can be smoothed together on the overlap regions in such a way that the Lyapunov properties are preserved.

Lemma 2. *For any $c > \frac{1}{a}$, $v_1(x, y) = x^2 + y^2$ is a local Lyapunov function on $\mathcal{R}_1$.*

Proof. v_1 clearly satisfies the first two properties of a local Lyapunov function, so it only remains to show the third property. Applying the generator to v_1, we obtain

$$(\mathcal{L}v_1)(x, y) = 2nx^2[(h'(x^m y^n)x^{m-1}y^{n-1}) - (h'(x^m y^n)x^{m-1}y^{n-1})^2] + \epsilon_x^2$$
$$+ 2my^2[-(h'(x^m y^n)x^{m-1}y^{n-1}) - (h'(x^m y^n)x^{m-1}y^{n-1})^2] + \epsilon_y^2.$$

Setting $u = |h'(x^m y^n)||x|^{m-1}|y|^{n-1}$, we have

$$(\mathcal{L}v_1)(x, y) \leq 2(nx^2 + my^2)(u - u^2) + \epsilon_x^2 + \epsilon_y^2.$$

Now $u - u^2$ is negative and strictly decreasing for $u > 1$. Recall that the constant $a > 0$ defined in Section 2 is a lower bound for $|h'|$, and hence, $|h'(x^m y^n)| \geq a$ for all (x, y). Thus, we can choose $c > \frac{1}{a}$ so that $u - u^2 \leq ac - (ac)^2 < 0$ for all $(x, y) \in \mathcal{R}_1$. This bound ensures that $\mathcal{L}v_1$ converges to negative infinity, and therefore v_1 is a local Lyapunov function on $\mathcal{R}_1$. ■

On the diffusive regions $\mathcal{R}_2$ and $\mathcal{R}_3$, we construct the local Lyapunov functions v_2 and v_3 as solutions to a boundary-value problem of the form

$$\begin{cases} (\tilde{\mathcal{L}}_i v_i)(x, y) = w_i(x, y) & \text{for } (x, y) \in \mathcal{R}_i \\ v_i(x, y) = \tilde{v}_i(x, y) & \text{for } (x, y) \in \partial\mathcal{R}_i, \end{cases} \quad (9)$$

where $\tilde{\mathcal{L}}_i$ consists of the terms in the generator $\mathcal{L}$ that scale dominantly in the region $\mathcal{R}_i$, w_i is chosen so that $\lim_{r\to\infty}\left[\sup_{(x,y)\in(\mathcal{R}_i\cap B_r^c)} w_i(x,y)\right] = -\infty$, and $\tilde{v}_i$ is asymptotic to v_1 on the boundary of $\mathcal{R}_i$. This method can be viewed as "propagating" an obvious Lyapunov function to regions of the plane where a Lyapunov function is not obvious.

In $\mathcal{R}_2$, the dominant term in the generator is $\tilde{\mathcal{L}}_2 = \frac{\epsilon_y^2}{2}\frac{\partial^2}{\partial y^2}$. For simplicity, we choose $\tilde{v}_2(x,y) = x^2$ and $w_2(x,y) = -k_2\epsilon_y^2 x^2$ with $k_2 > 0$. Likewise, in $\mathcal{R}_3$, the dominant term in the generator is $\tilde{\mathcal{L}}_3 = \frac{\epsilon_x^2}{2}\frac{\partial^2}{\partial x^2}$, and we choose $\tilde{v}_3(x,y) = y^2$ and $w_3(x,y) = -k_3\epsilon_x^2 y^2$ with $k_3 > 0$. With these choices, we can find explicit solutions to the boundary-value problem described by (9), which are given in the lemma below.

Lemma 3. *For any $c > 0$,*

$$v_2(x,y) = x^2(1 - k_2 y^2)$$

with $k_2 > \frac{n}{2\epsilon_y^2}$ is a local Lyapunov function on $\mathcal{R}_2$ and

$$v_3(x,y) = y^2(1 - k_3 x^2)$$

with $k_3 > \frac{m}{2\epsilon_x^2}$ is a local Lyapunov function on $\mathcal{R}_3$.

Proof. v_2 and v_3 clearly satisfy the first two properties of a local Lyapunov function on their respective regions since $\mathcal{R}_2$ consists of a decaying strip about the x-axis and $\mathcal{R}_3$ consists of a decaying strip about the y-axis. Applying the generator to v_2 we obtain

$$\begin{aligned}(\mathcal{L}v_2)(x,y) &= 2nx^2(1-k_2y^2)[(h'(x^my^n)x^{m-1}y^{n-1}) - (h'(x^my^n)x^{m-1}y^{n-1})^2] \\ &\quad - 2mk_2x^2y^2[-(h'(x^my^n)x^{m-1}y^{n-1}) - (h'(x^my^n)x^{m-1}y^{n-1})^2] \\ &\quad + \epsilon_x^2(1-k_2y^2) - \epsilon_y^2k_2x^2.\end{aligned}$$

Setting $u = |h'(x^my^n)||x|^{m-1}|y|^{n-1}$, we have

$$\begin{aligned}(\mathcal{L}v_2)(x,y) &\le 2nx^2(u-u^2) + 2k_2(n+m)x^2y^2(u+u^2) \\ &\quad + \epsilon_x^2(1-k_2y^2) - \epsilon_y^2k_2x^2.\end{aligned}$$

Since $\mathcal{R}_2$ consists of the decaying strip around the x-axis, the terms with x^2 will dominate in $\mathcal{L}v_2$. Hence, we need to ensure that the net sign of the coefficient of the x^2 terms is negative. Now the maximum of $u - u^2$ occurs at $u = \frac{1}{2}$ and is equal to $\frac{1}{4}$. Thus, if we choose $k_2 > \frac{n}{2\epsilon_y^2}$, then $\mathcal{L}v_2$ will converge to negative infinity in $\mathcal{R}_2$. The proof that $\mathcal{L}v_3$ converges to negative infinity in $\mathcal{R}_3$ is analogous. ∎

Remark. Due to the nature of the deterministic dynamics, it is the noise in the y-direction, $\epsilon_y \neq 0$, that is crucial to the existence of a local Lyapunov function in $\mathcal{R}_2$ and it is the noise in the x-direction, $\epsilon_x \neq 0$, that is crucial to the existence of a local Lyapunov function in $\mathcal{R}_3$. Thus, in order to obtain a global Lyapunov function on the entire plane, noise is needed in both the x- and y-directions.

Since we have shown the existence of local Lyapunov functions on regions covering the entire plane, minus some ball about the origin, we now seek to patch them together to form one smooth, global Lyapunov function. Since the local regions overlap, the straightforward approach is to construct convex combinations of the form

$$v_{ij}(x, y) = \phi(r(x, y))v_i(x, y) + (1 - \phi(r(x, y)))v_j(x, y)$$

on the overlap regions such that $\phi(r(x, y))$ is a smooth function with $\phi(r(x, y)) = 0$ on one border and $\phi(r(x, y)) = 1$ on the other. In particular, we set

$$r(x, y) = \frac{|x|^{m-1}|y|^{n-1} - c}{c} \quad \text{and}$$

$$\phi(t) = \frac{\int_{-\infty}^{t} \psi(s)ds}{\int_{-\infty}^{\infty} \psi(s)ds} \quad \text{where} \quad \psi(t) = \begin{cases} \exp\left(\frac{-1}{1-(2t-1)^2}\right) & \text{for } 0 < t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

The mollifier $\phi(r(x, y))$ ensures that the convex combinations v_{ij} satisfy the first two local Lyapunov properties (i.e., the smoothness and growth conditions) on the appropriate overlap regions. However, it is not guaranteed that the convex combinations will satisfy the third local Lyapunov property since additional terms result after the application of the generator. In Lemma 4, we prove that for appropriate choice of constants, our convex combinations are indeed local Lyapunov functions on the overlap regions.

Lemma 4. *There exists $c > \frac{1}{a}$ such that for any $k_2 > \frac{n}{2\epsilon_y^2}$,*

$$v_{12}(x, y) = \phi(r(x, y))v_1(x, y) + (1 - \phi(r(x, y)))v_2(x, y)$$

is a local Lyapunov function on

$$\mathcal{R}_1 \cap \mathcal{R}_2 = \{(x, y) : |x| \geq 1, c \leq |x|^{m-1}|y|^{n-1} \leq 2c\},$$

and for any $k_3 > \frac{m}{2\epsilon_x^2}$,

$$v_{13}(x, y) = \phi(r(x, y))v_1(x, y) + (1 - \phi(r(x, y)))v_3(x, y)$$

is a local Lyapunov function on

$$\mathcal{R}_1 \cap \mathcal{R}_3 = \{(x, y) : |y| \geq 1, c \leq |x|^{m-1}|y|^{n-1} \leq 2c\}.$$

Proof. Applying the generator to v_{12} we obtain

$$\begin{aligned}(\mathcal{L}v_{12})(x, y) = {} & \phi(r(x, y))(\mathcal{L}v_1)(x, y) + (1 - \phi(r(x, y)))(\mathcal{L}v_2)(x, y) \\ & + \mathcal{L}[\phi(r(x, y))](v_1(x, y) - v_2(x, y)) \\ & + \epsilon_x^2 \frac{\partial}{\partial x}[\phi(r(x, y))] \frac{\partial}{\partial x}[v_1(x, y) - v_2(x, y)] \\ & + \epsilon_y^2 \frac{\partial}{\partial y}[\phi(r(x, y))] \frac{\partial}{\partial y}[v_1(x, y) - v_2(x, y)].\end{aligned}$$

From the proofs of Lemmas 2 and 3, on $\mathcal{R}_1 \cap \mathcal{R}_2$ with $c > \frac{1}{a}$,

$$\phi(r(x, y))(\mathcal{L}v_1)(x, y) + (1 - \phi(r(x, y)))(\mathcal{L}v_2)(x, y) \leq D(x, y),$$

where $D(x, y) \sim 2n(ac - a^2c^2)x^2$. From explicit computation, the remaining terms in $\mathcal{L}v_{12}$ are less than or equal to $E(x, y)$ on $\mathcal{R}_1 \cap \mathcal{R}_2$, where

$$E(x, y) \sim k_2\epsilon_y^2(8(n-1) + 2(n-1)(n-2) + 16(n-1)^2)x^2.$$

The above asymptotic expression for $E(x, y)$ results from using the fact that $|\phi'(t)| \leq 2$ and $|\phi''(t)| \leq 8$ for all t. Hence,

$$\begin{aligned}(\mathcal{L}v_{12})(x, y) &\leq D(x, y) + E(x, y) \\ &\sim [2n(ac - a^2c^2) + k_2\epsilon_y^2(8(n-1) + 2(n-1)(n-2) + 16(n-1)^2)]x^2.\end{aligned}$$

Since the coefficient of x^2 is a quadratic function of c with leading term $-2na^2c^2$, we can choose c sufficiently large so that the coefficient of x^2 is negative. This result then implies that $\mathcal{L}v_{12}$ converges to negative infinity in $\mathcal{R}_1 \cap \mathcal{R}_2$. The proof for $\mathcal{L}v_{13}$ in $\mathcal{R}_1 \cap \mathcal{R}_3$ is analogous. ■

Our global Lyapunov function, $V(x, y) \in C^\infty(\mathbb{R}^2)$, can now be constructed so that

$$V(x, y) = \begin{cases} \tilde{V}(x, y) & \text{for } x^2 + y^2 > \rho^2 \\ \text{arbitrary positive and smooth} & \text{for } x^2 + y^2 \leq \rho^2, \end{cases}$$

where $\rho \geq 4c$ and

$$\tilde{V}(x, y) = \begin{cases} v_1(x, y) & \text{for } (x, y) \in \mathcal{R}_1 \cap \mathcal{R}_2^c \cap \mathcal{R}_3^c \\ v_2(x, y) & \text{for } (x, y) \in \mathcal{R}_1^c \cap \mathcal{R}_2 \cap \mathcal{R}_3^c \\ v_3(x, y) & \text{for } (x, y) \in \mathcal{R}_1^c \cap \mathcal{R}_2^c \cap \mathcal{R}_3 \\ v_{12}(x, y) & \text{for } (x, y) \in \mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_3^c \\ v_{13}(x, y) & \text{for } (x, y) \in \mathcal{R}_1 \cap \mathcal{R}_2^c \cap \mathcal{R}_3. \end{cases}$$

The existence of this global Lyapunov function implies that the perturbed Hamiltonian system with noise is indeed stable for any $\epsilon_x \neq 0$ and $\epsilon_y \neq 0$ and completes the proof of Theorem 1.

4. CONCLUDING REMARKS.

While in this article we have shown how to deterministically perturb a class of Hamiltonian systems so that they exhibit noise-induced stabilization, the deterministic perturbation that we constructed is certainly not unique. For example, our $\frac{dx}{dt}$ perturbation had the form

$$-nx(h'(x^m y^n)x^{m-1}y^{n-1})^2$$

but any perturbation of the form

$$-nx(h'(x^m y^n)x^{m-1}y^{n-1})^q$$

with q a positive even integer, along with a corresponding $\frac{dy}{dt}$ perturbation, would also result in enabling noise-induced stabilization. An interesting open question is whether there exists in some rigorous sense a "minimal" perturbation that would most closely preserve the behavior of the original Hamiltonian system, but allow for noise to have a stabilizing effect.

Furthermore, in this article we have only considered noise with constant coefficients, but one could also explore allowing the strength of the noise to depend upon space, such as in [1,3]. In the case of nonconstant noise, it may be possible for the class of Hamiltonian systems to exhibit noise-induced stabilization without a deterministic perturbation.

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