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4−EQUITABLE TREE LABELINGS

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Abstract. We assign the labels $\{0, 1, 2, 3\}$ to the vertices of a graph; each edge is assigned the absolute difference of the incident vertices' labels. For the labeling to be 4−equitable, we require the edge labels and vertex labels to each be distributed as uniformly as possible. We study 4−equitable labelings of different trees and prove all caterpillars, symmetric generalized n−stars (or symmetric spiders), and complete n−ary trees for all $n \in \mathbb{N}$ are 4−equitable.

In 1964 Ringel conjectured that given any tree on n vertices, the edges of the complete graph on $2n + 1$ vertices, K_{2n+1}, can be decomposed into isomorphic copies of this tree [Rin64]. In order to attack this conjecture, Rosa introduced certain labelings of the vertices and edges of trees. The most famous of these labelings, the β−valuations, were renamed graceful labelings by Golomb [Gol72]. In a graceful labeling of a graph with e edges, each vertex gets a distinct label from the set $\{0, 1, \ldots, e\}$ so that each edge (labeled using the absolute difference of its incident vertices), has a unique label [Ros67].

Rosa proved that if a tree has a graceful labeling then Ringel's decomposition conjecture is true for that tree [Ros67]. Based on this theorem Ringel and Kotzig conjectured that all trees were graceful. This conjecture is now known as the graceful tree conjecture, and it is arguably the most famous graph labeling conjecture.

In the past 50 or so years a whole field of graph theory grew from this central conjecture. There have been many generalizations and some partial results, but we are not much closer today to proving the conjecture than we were fifty years ago. An excellent and thorough survey of all results related to the graceful tree conjecture can be found in Gallian's *Dynamic Survey of Graph Labeling* [Gal16].

In particular, in 1995, Cahit generalized the concept of graceful labelings to k−equitable labelings. In a k−equitable labeling the vertex labels come

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from the set $\{0, 1, \ldots, k-1\}$. The edge labels are defined the same way as in a graceful labeling, the absolute difference of the incident vertex labels. The labeling is $k-$equitable if both the vertex labels and the edge labels are distributed as evenly as possible. The formal definition is as follows:

Definition 1. *Let $G(V, E)$ be a graph with vertex set V and edge set E. Given $f : V \to 0, 1, 2, \ldots, k-1$ we define $f(e) = |f(u) - f(v)|$ for all $e = (u, v) \in E$. Let v_i and e_i denote the number of vertices, respectively edges, labeled i. We say that f is a $k-$equitable labeling of G if $|v_i - v_j| \leq 1$ and $|e_i - e_j| \leq 1$ for every i, j*

Notice that if a tree has size e then the definitions of $(e-1)-$equitability and graceful coincide. Therefore, if one could show that every tree is $k-$equitable for every k then the proof would settle the graceful tree conjecture as well. Unfortunately, there is very little known about $k-$equitability of trees for large k values. Cahit conjectured that all trees are $k-$equitable, [Cah90a]. He proved that all trees were $2-$equitable (also known as cordial), and some trees were $3-$equitable, [Cah90b]. Speyer and Szaniszlo later proved that all trees were $3-$equitable [SS00].

In this paper we consider $4-$equitable labelings of different classes of trees. We show that caterpillars, symmetric generalized $n-$stars (or spiders), and complete $n-$ary trees are $4-$equitable.

Throughout the paper we will refer to a vertex labeled i as an *$i-$vertex*. Likewise, we will refer to an edge labeled i as an *i-edge*.

1. Caterpillars

Definition 2. *A* caterpillar *is a tree where every vertex is at most distance one away from the longest path. This longest path is called the* spine. *Any leaves not on the spine are called* legs.

Lemma 1. *There exists a $4-$equitable labeling of the spine where there are at least as many legs connected to $0-$ and $3-$vertices as there are connected to $1-$and $2-$vertices.*

Proof. Given any caterpillar, find its spine. Label this path with the pattern $3-0-2-1-1-2-0-3$, repeating as necessary. If the original path labeling led to more legs from $1-$ and $2-$vertices than from $0-$ and $3-$vertices, then change to the repeating pattern $1-2-0-3-3-0-2-1$. Since the second pattern switches every 1 and 2 with 3 and 0 respectively, we know that it is possible to have at least as many legs adjacent to $0-$ and $3-$vertices as to $1-$ and $2-$vertices. $\square$

Theorem 1. *All Caterpillars are $4-$equitable.*

Proof. We label the caterpillar in steps, ensuring we preserve the $4-$equitability at each step. Because the only way to create a $3-$edge is by connecting a $0-$vertex and a $3-$vertex, we need to have enough legs that are adjacent to such vertices on the spine. Label the caterpillar's spine so that there are no fewer legs from $0-$ and $3-$vertices than $1-$ and $2-$vertices. If the spine is of length $1 \pmod 4$ or of length $3 \pmod 4$, do not label the last vertex. In these two cases, we treat the unlabeled edge and vertex as a leg, and not as part of the spine. This means paths of length $0 \pmod 4$ and $1 \pmod 4$ will be treated in the same manner, as will paths of length $2 \pmod 4$ and $3 \pmod 4$. We will modify the original spine labeling later as necessary.

Once we have labeled the spine to have at least as many legs from $0-$ and $3-$ vertices as $1-$ and $2-$vertices, the actual location of the legs is irrelevant, we just need to note what types of vertices the remaining unlabeled vertices are adjacent to. The next step is to label the legs in groups of four, ensuring we create four distinct vertex labels and four distinct edge labels until we can no longer do so . From the list below, we repeat step i as many times as possible before moving to step $i+1$. This is not the only possible ordering of the labeling steps, but we need to be careful not to use up too many of the legs connected to $0-$vertices and $3-$vertices in any given step.

(1) If there is an unlabeled leg available on each vertex labeled 0, 1, 2 and 3, then create edges $0-3$, $1-0$, $2-2$, and $3-1$
(2) If there are 2 unlabeled legs available on $0-$vertices and 2 on $2-$vertices, then create edges $0-3$, $0-1$, $2-2$, and $2-0$.
(3) If there are 2 unlabeled legs available on $2-$vertices and 2 on $3-$vertices, then create edges $2-3$, $2-2$, $3-1$, and $3-0$.
(4) If there are 2 unlabeled legs available on $0-$vertices and 2 on $1-$vertices, then create edges $0-3$, $0-2$, $1-1$, and $1-0$.
(5) If there are 2 unlabeled legs available on $1-$vertices and 2 on $3-$vertices, then create edges $1-3$, $1-1$, $3-0$, and $3-2$.
(6) If there is 1 unlabeled leg available on a $2-$vertex and 3 on $0-$vertices, then create edges $0-0$, $0-2$, $0-3$, and $2-1$.
(7) If there is 1 unlabeled leg available on a $2-$vertex, 1 on a $0-$vertex, and 2 on $3-$vertices, then create edges $2-1$, $0-2$, $3-3$, and $3-0$.
(8) If there is 1 unlabeled leg available on a $1-$vertex and 3 on $3-$vertices, then create edges $3-0$, $3-1$, $3-3$, and $1-2$.
(9) If there is 1 unlabeled leg available on a $1-$vertex, 1 on a $3-$vertex, and 2 on $0-$vertices, then create edges $1-2$, $3-1$, $0-3$, and $0-0$.
(10) If there are 2 unlabeled legs available on $0-$vertices and 2 on $3-$vertices, then create edges $0-0$, $0-3$, $3-1$, and $3-2$.
(11) If there are 4 unlabeled legs available on $0-$vertices, then create edges $0-0$, $0-1$, $0-2$, and $0-3$.

(12) If there are 4 unlabeled legs available on $3-$vertices, then create edges $3-0$, $3-1$, $3-2$, and $3-3$.

We have labeled legs in groups of 4, adding 4 distinct vertex labels and 4 distinct edge labels in each step. This way, the $4-$equitability of the caterpillar is the same as the $4-$equitability of the spine. Hence the distribution of extra edge and vertex labels does not change.

If the spine was of length $0 \pmod 4$, either path labeling pattern results in a perfectly even set of vertex labels, and one less $0-$edge. If the spine was of length $2 \pmod 4$, using the pattern results in either an extra $0-$ and $3-$vertex, and an extra $3-$edge, or results in an extra $1-$ and $2-$vertex, and an extra $1-$edge.

After reducing the amount of unlabeled legs as much as possible by the previous groupings of 4, we will be left with up to 4 unlabeled vertices. The only way we can be left with more than 2 legs from a $1-$vertex or from a $2-$vertex is if we have: 2 legs from $1-$vertices, 1 from a $0-$vertex, and 1 from a $3-$vertex, or 2 legs from $2-$vertices, 1 from a $0-$vertex, and 1 from a $3-$vertex. Otherwise we have up to 1 unlabeled leg from a $1-$vertex, up to 1 from a $2-$vertex, up to 3 from $0-$vertices, and up to 3 from $3-$vertices. Unfortunately, even though there are no more than 4 vertices to label, we will need to consider over 20 possibilities. In the rest of this section, we will present all of these for the sake of completeness.

Case 1. *The spine of the caterpillar is of length* $0 \pmod 4$.

Notice that in this case we have a missing $0-$edge in both labelings of the spine, so we will need to create a $0-$edge before creating different edges. Since the vertex labels on the spine are uniformly distributed all our new vertex labels must be different.

(1) If there is one unlabeled leg from a $2-$vertex, one from a $1-$vertex and two from $0-$vertices, create the edges $1-1$, $2-3$, $0-2$, $0-0$.
(2) If there is one unlabeled leg from a $2-$vertex, one from a $1-$vertex and two from $3-$vertices, create the edges $2-2$, $1-0$, $3-1$, $3-3$.

Otherwise: if there exists a leg on a $1-$vertex, create a $0-$edge by labeling the leaf 1. If there exists a leg on a $2-$vertex, create a $0-$edge by labeling the leaf 2. If all legs are on $0-$ and $3-$vertices, create edge $0-0$ and/or $3-3$, then use other vertex labels to complete the labeling.

Case 2. *The spine of the caterpillar is of length* $2 \pmod 4$*, with an extra* $3-$*edge.*

This happens when we use the first spine labeling pattern. In this case we have an extra $0-$vertex and an extra $3-$vertex, as well as an extra $3-$edge. Therefore we must first use 1 and 2 as vertex labels before using any other vertex label.

If there is exactly one unlabeled leg on both a $0-$vertex and a $3-$vertex, we will need to go back to the labeling of the caterpillar's spine, which we labeled with the pattern $3-0-2-1-1-2-0-3$. In order to $4-$equitably label the graph, we will remove the label on the first $3-$vertex on the spine, and relabel it with a 1. Notice that although this is a spine vertex, it has degree 1, so we only change one edge label in this process. We now have 1 extra $1-$edge, $0-$vertex, and $1-$vertex. Then, we will use the unlabeled legs to create edges $0-2$ and $3-3$, which will result in a $4-$equitable labeling.

Otherwise, we can define the labeling based on what types of vertices the unlabeled legs are connected to as follows:

(1) If there are only unlabeled legs from $0-$vertices, create edges in the order $0-1$, $0-2$, then $0-0$ as necessary.
(2) If there are only unlabeled legs from $3-$vertices, create edges in the order $3-1$, $3-2$, then $3-3$ as necessary.
(3) If there is one unlabeled leg from a $3-$vertex, and the rest (at least two) are from $0-$vertices, create the edges in the order $3-3$, $0-1$, $0-2$, $0-0$.
(4) If there is one unlabeled leg from a $0-$vertex, and the rest (at least two) are from $3-$vertices, create the edges in the order $0-0$, $3-1$, $3-2$, $3-3$.
(5) If there is one unlabeled leg from a $2-$vertex, and the rest (if any) are from $0-$vertices, create the edges in the order $2-1$, $0-2$, $0-0$.
(6) If there is one unlabeled leg from a $2-$vertex, and the rest are from $3-$vertices, create the edges in the order $2-2$, $3-1$, $3-2$, $3-3$.
(7) If there is one unlabeled leg from a $2-$vertex, one from a $3-$vertex and the rest are from $0-$vertices, create the edges in the order $2-2$, $3-1$, $0-1$, $0-2$.
(8) If there is one unlabeled leg from a $1-$vertex, and the rest are from $0-$vertices, create the edges in the order $1-1$, $0-2$, $0-1$, $0-0$.
(9) If there is one unlabeled leg from a $1-$vertex, and the rest (if any) are from $3-$vertices, create the edges in the order $1-2$, $3-1$, $3-3$.
(10) If there is one unlabeled leg from a $0-$vertex, one from a $1-$vertex and the rest are from $3-$vertices, create the edges in the order $0-0$, $1-2$, $3-1$, $3-2$.
(11) If there is one unlabeled leg from a $2-$vertex, one from a $1-$vertex and two from $0-$vertices, create the edges $1-3$, $2-2$, $0-1$, $0-2$.
(12) If there is one unlabeled leg from a $2-$vertex, one from a $1-$vertex and two from $3-$vertices, create the edges $2-2$, $1-0$, $3-1$, $3-2$.

Case 3. *The length of the spine is of* $2 \pmod 4$*, with an extra* $1-$*edge.*

This happens when we use the second spine labeling pattern. In this case we have an extra $1-$vertex and an extra $2-$vertex. Therefore we must use 0 and 3 as vertex labels before using any other vertex label.

If there is exactly one unlabeled leg on both a $0-$vertex and a $3-$vertex, we will need to go back to the labeling of the caterpillar's spine, which we labeled with the pattern $1-2-0-3-3-0-2-1$. In order to $4-$equitably label the graph, we will remove the label on the first $1-$vertex on the spine, and relabel it with a 0. Then, we will use the unlabeled legs to create edges $0-1$ and $3-3$, which will result in a $4-$equitable labeling. Otherwise, we can define the labeling based on what types of vertices the unlabeled legs are connected to as follows:

(1) If there are only unlabeled legs from $0-$vertices, create edges in the order $0-3$, $0-0$, then $0-2$ as necessary.
(2) If there are only unlabeled legs from $3-$vertices, create edges in the order $3-0$, $3-3$, then $3-1$ as necessary.
(3) If there is one unlabeled leg from a $3-$vertex, and the rest (at least two) are from $0-$vertices, create the edges in the order $3-1$, $0-0$, $0-3$, $0-2$.
(4) If there is one unlabeled leg from a $0-$vertex, and the rest (at least two) are from $3-$vertices, create the edges in the order $0-2$, $3-0$, $3-3$, $3-1$.
(5) If there is one unlabeled leg from a $2-$vertex, and the rest (if any) are from $0-$vertices, create the edges in the order $2-0$, $0-3$, $0-0$.
(6) If there is one unlabeled leg from a $2-$vertex, and the rest are from $3-$vertices, create the edges in the order $2-0$, $3-3$, $3-0$, $3-1$.
(7) If there is one unlabeled leg from a $2-$vertex, one from a $3-$vertex and the rest are from $0-$vertices, create the edges in the order $2-0$, $3-3$, $0-3$, $0-1$.
(8) If there is one unlabeled leg from a $1-$vertex, and the rest are from $0-$vertices, create the edges in the order $1-3$, $0-0$, $0-3$, $0-1$.
(9) If there is one unlabeled leg from a $1-$vertex, and the rest (if any) are from $3-$vertices, create the edges in the order $1-3$, $3-0$, $3-3$.
(10) If there is one unlabeled leg from a $0-$vertex, one from a $1-$vertex and the rest are from $3-$vertices, create the edges in the order $0-0$, $1-3$, $3-0$, $3-3$.
(11) If there is one unlabeled leg from a $2-$vertex, one from a $1-$vertex and two from $0-$vertices, create the edges $1-1$, $2-0$, $0-3$, $0-2$.
(12) If there is one unlabeled leg from a $2-$vertex, one from a $1-$vertex and two from $3-$vertices, create the edges $2-2$, $1-3$, $3-0$, $3-1$.

$\square$

The fact that we needed so many cases to handle the last few remaining vertices suggests that this method is not easily generalizable for higher

$k-$values when establishing the $k-$equitability of caterpillars. However, we are able to conclude that all caterpillars are $4-$equitable.

2. Symmetric Generalized $n-$stars

Definition 3. *A* symmetric generalized $n-$star *for* $n \geq 3$ *is a tree in which all leaves are the same distance from a central vertex of degree* n*. The number of leaves in the tree is equal to the maximum degree of the graph. A* level *is the set of all vertices the same distance away from the degree* n *vertex.*

The concept of a level makes intuitive sense when we think about these graphs as rooted trees with the high degree vertex being the root. We also note here that we could have used the name symmetric spider, after spider graphs defined in the literature, instead of symmetric generalized $n-$star, but we find the latter more descriptive even if somewhat less imaginative.

Lemma 2. *All symmetric generalized* $3-$*stars are* $4-$*equitable.*

Proof. Let us consider a symmetric generalized $3-$star as a tree rooted at the vertex of maximum degree. We use a labeling pattern where we label each level at a time such that the tree stays $4-$equitable at each step of the pattern. To do this, we use the $4-$subsets of $\{0, 1, 2, 3\}$ with a cardinality of three as vertex labels and label the vertices in a way such that the edge differences also produce the $4-$subsets of $\{0, 1, 2, 3\}$. We start by labeling the root vertex 0. We then go in a clockwise direction following the pattern $3-2-1, 3-0-2, 0-1-2, 3-1-0$. This takes care of the first four levels of the symmetric generalized 3-star. For the next four levels, since we are not starting from a root vertex of zero, but are continuing from our $3-1-0$ leaves, we must start from the beginning, switching path one with path three and keeping the center path the same. Namely, the vertex labels on the next four levels of the complete generalized $3-$star are $1-2-3, 2-0-3, 2-1-0, 0-1-3$ moving in the same direction. These two patterns alternate as we label further and further along a symmetric generalized $3-$star. At each level, the vertex labels produce three different edge labels using three different vertex labels, and the star remains $4-$equitable at each level. □

Lemma 3. *All symmetric generalized* $4-$*stars are* $4-$*equitable.*

Proof. With a similar argument, we can also see that all symmetric generalized $4-$stars are also 4-equitable. We begin by labeling the root vertex 0. We then label, following a clockwise motion, the vertices at each level with $0-1-2-3$, then $2-1-3-0$. Since we are working with exactly four vertices at each level, we can simply alternate between these two labelings.

What is essential is that we use four different vertex labels and four different edge labels at each level. Thus, the tree remains $4-$equitable as the symmetric generalized 4-star gets larger and larger. Specifically, the edges are distributed evenly and there is one extra zero vertex. $\square$

Lemma 4. *All symmetric generalized* $5-$*stars are* $4-$*equitable.*

Proof. Notice that we can add a path to a symmetric generalized $4-$star to create a symmetric generalized $5-$star. In order to keep the new generalized $5-$star $4-$equitable at each level, we must choose a vertex label that will be repeated at each level as well as an edge label to repeated at each level. To do this, we can simply label this extra path following the path labeling pattern that starts with 0 that is already the label of the root vertex and we continue with $3-1-2-2-1-3-0$ which we know produces different edge labels and vertex labels at each step in groups of 4, which is consistent with the labels for the complete generalized $4-$star. This results in a $4-$equitable labeling of the symmetric generalized star with five leaves. $\square$

Lemma 5. *All symmetric generalized* $6-$*stars are* $4-$*equitable.*

Proof. With a similar argument, we can add two paths to the symmetric generalized $4-$star to create a symmetric generalized $6-$star. Notice in the alternate path labeling pattern : $3-0-2-1-1-2-\underline{\mathbf{0}}-3-3-0-2-1-1$ that the vertices and edges are asymmetric about the indicated 0. We will take this vertex label as the label of the root vertex. The asymmetry of the path labeling pattern about the root vertex is essential for keeping the symmetric generalized $6-$star $4-$equitable at each step. $\square$

Theorem 2. *All symmetric generalized stars are* $4-$*equitable.*

Proof. To create a $4-$equitable labeling of a symmetric generalized $n-$star for all $n \geq 3$ we will glue copies of a generalized $4-$star with one copy of a symmetric generalized $m-$star where $m < 4$ at the root. So, if $n = 4k + m$ and $m \neq 1, 2$ then we glue k copies of our labeled symmetric generalized $4-$star and one copy of our labeled symmetric generalized $m-$star. If $m = 1$ or 2, we use $k - 1$ copies of the symmetric generalized $4-$star and one copy of our symmetric generalized $5-$ or $6-$star, respectively.

$\square$

3. Complete $n-$ary Trees

In order to prove all complete $n-$ary trees are $4-$equitable, we first handle binary, ternary, quaternary, and $5-$ary, $6-ary$ and 7-ary trees. We will build the general case from these building blocks.

Lemma 6. *All Complete Binary Trees are* $4-$*equitable.*

Proof. Given any complete binary tree, label the root vertex 3. Label the children of each $0-$ and $2-$vertices 2 and 3; label the children of each $1-$ and $3-$vertices 0 and 1. On level 2 the 4 vertices are labeled $0, 1, 2, 3$. The number of vertices on every future level is a multiple of 4, and the vertex labels are evenly distributed on each level. The edges between any $3-$ or $0-$vertex and its children are 3 and 2; the edges between any $2-$ or $1-$vertex and its children are 0 and 1. The edge labels between levels 1 and 2 are completely evenly distributed. The number of edges after every future level is a multiple of 4, and the edge labels are evenly distributed on each level. Therefore, all complete binary trees are $4-$equitable. □

In order to prove that complete ternary trees are $4-$equitable we need to carefully count the number of vertices and edges in the tree level by level. Since the number of edges at each level is a power of three, we need to know how we can write powers of three modulo 4. We will make use of the following two easy facts:

(1) If a number has the form $(8a+3)$, then $9(8a+3) = (8+1)(8a+3) = 64a + 8a + 24 + 3 = 8B + 3$. To obtain an odd power of 3 we can multiply 3 with 9 as many times as necessary. Therefore, every odd exponent of 3 is of the form $(8B+3)$ and is congruent to $3 \pmod 4$.

(2) If a number has the form $(8a+1)$, then $9(8a+1) = (8+1)(8a+1) = 8B+1$. Since every even exponent of 3 is a product of 9's this implies that every even exponent of 3 is of the form $8B+1$ and is congruent to $1 \pmod 4$.

Now we establish how the total number of vertices in a complete ternary tree with k levels, $\frac{3^{k+1}-1}{2}$, relates to 4.

Claim 1. $\frac{3^{k+1}-1}{2} \equiv 1 \pmod 4$ *when k is even, and* $\equiv 0 \pmod 4$ *when k is odd.*

Proof. If k is even, then $k+1$ is odd, so $3^{(k+1)}$ is of the form $8B+3$, and $\frac{8B+3-1}{2} = 4B + 1 \equiv 1 \pmod 4$. If k is odd, then the exponent is even, so 3^{k+1} is of the form $8B + 1$, and $8B + 1 - 1 = 8B$, which is divisible by 4. □

Lemma 7. *All complete ternary trees are $4-$equitable.*

Proof. Given any complete ternary tree, label the root vertex 3. Label the children of each $3-$vertex with $0, 1$ and 2. Label the children of each $2-$vertex, with $0, 1$ and 2. Label the children of each $1-$vertex, with $1, 2$ and 3. Label the children of each $0-$vertex with $3, 3$ and 0. The total number of vertices in the tree is given by $\sum_{i=0}^{k} 3^i = \frac{3^{k+1}-1}{2}$. This sum is congruent to either $1 \pmod 4$ or $0 \pmod 4$. Thus we know that either the number of vertices or number of edges in the tree is divisible by 4.

Let $v_{l,i}$ be the number of vertices labeled i in level l, and $e_{l,i}$ be the number of edges labeled i in level l. The distribution of vertex labels in level $l+1$ is as follows: The number of $0-$vertices in level $l+1$ is equal to $v_{l,0}+v_{l,2}+v_{l,3}$; the number of $1-$vertices in level $l+1$ is equal to $v_{l,1}+v_{l,2}+v_{l,3}$; The number of $2-$vertices in level $l+1$ is equal to $v_{l,1}+v_{l,2}+v_{l,3}$; and the number of $3-$vertices in level $l+1$ is equal to $2v_{l,0}+v_{l,2}$. In level 1 we have $v_{1,3}=0$ and $v_{1,1}=v_{1,2}=v_{1,0}=1$ and, with the above formulas, we can see that there is an extra $3-$vertex in level 2. Substituting $v_{2,0}$ into the previous formulas we get $v_{3,0}=v_{3,1}=v_{3,2}=v_{2,0}+v_{2,0}+(v_{2,0}+1)=3v_{1,0}+1$. Also, $v_{3,3}=2v_{2,0}+v_{1,0}=3v_{2,0}$, so level 3 has one less $3-$vertex than $0-$, $1-$, and $2-$vertices. Since every odd exponent of 3 is congruent to 3 (mod 4), and every even exponent of 3 is congruent to 1(mod 4), the pattern of having one extra $3-$vertex in a level and one extra 0, 1, and $2-$vertices in the next level continues to repeat infinitely. This means if the vertex labels include an extra 3 in one level, then the labels of the next level include an extra 0, 1, and 2, and the reverse also holds true. Furthermore, because the total number of vertices in the tree is congruent to either 1(mod 4) or 0(mod 4), the pattern for total vertices in the tree of an extra $3-$vertex and then completely even vertex labels repeats infinitely. A similar argument for the total number of edge labels shows a repeating pattern of one less $0-$edge followed by the edge labels being distributed completely evenly in the whole graph. Therefore, all complete ternary trees are $4-$equitible. □

In what follows we will describe labeling algorithms for other $n-$ary trees. We will leave it to the reader to produce an argument similar to the above argument for checking the $k-$equitability at each level, as we feel it would not add to the clarity. Whenever possible we will introduce the same number of vertex labels of each kind and the same number of edges labels of each kind on any new level. This will assure that the property of being $4-$equitable is preserved in each step. The introduction of the following two terms will help simplify the the proofs.

Definition 4. *A group of* 4 *vertices each having distinct labels is called a* perfect group*. The children of a perfect group are labeled by the* perfect group algorithm *as follows: All the children of the* $0-$*vertex are labeled* 3*. All the children of the* $1-$*vertex are labeled* 0*. All the children of the* $2-$*vertex are labeled* 2*. All the children of the* $3-$*vertex are labeled* 1*.*

Notice that executing the perfect group algorithm introduces the same number of vertex and edge labels of each kind. Therefore we can apply this algorithm repeatedly and the the $4-$equitability of the tree will not change.

Lemma 8. *All complete quaternary trees are* $4-$*equitable.*

Proof. Given any complete quaternary tree, label the root vertex with 0. Label the children of the root vertex with $0, 1, 2$ and 3. Label all descendants using the perfect group algorithm. $\square$

Lemma 9. *All complete* $5-$*ary trees are* $4-$*equitable.*

Proof. Given any complete $5-$ary tree, label the root vertex 0. Label the vertices in level 1 by $0, 1, 2, 3$ and 3. Label the descendants of the perfect group by using the perfect group algorithm repeatedly.

Label the children of the $3-$vertex in level 1 with $0, 1, 1, 2$ and 3 . Label the children of the $0-$vertex by 3. Label the children of the $2-$vertex by 0. Label the children of one of the $1-$vertices by 2 and label the children of the other $1-$vertex by 1. Group these new vertices into perfect groups and label their descendants by using the perfect group algorithm repeatedly.

Label the children of the $3-$vertex on level 2 by $1, 2, 2, 3$ and 0. Label the children of the $1-$ vertex by 3. Label the children of the $3-$vertex by 0. Label the children of one of the $2-$vertices by 2 and label the children of the other $2-$vertex by 1. Group these new vertices into perfect groups and label their descendants by using the perfect group algorithm repeatedly.

Label the children of the $0-$vertex on level 3 by $0, 1, 2, 3$ and 0. Label the descendants of the perfect group by using the perfect group algorithm repeatedly. Treat the $0-$vertex on level 4 like the root vertex and repeat the above algorithm. A complete $5-$ary tree labeled by the above algorithm is equitable at any height. $\square$

Lemma 10. *All complete* $6-$*ary trees are* $4-$*equitable.*

Proof. Given any complete $6-$ary tree, label the root vertex 0. Label the vertices in level 1 by $1, 2, 2, 3$ and $3, 0$. First we define the labeling of the descendants of the first group of 4 vertices. Label the children of the $1-$vertex by 3. Label the children of the $3-$vertex by 0. Label the children of one of the $2-$vertices by 2 and label the children of the other $2-$vertex by 1. Group these new vertices into perfect groups and label their descendants by using the perfect group algorithm repeatedly.

For the two remaining vertices on level 1 label half of the children of the $3-$vertex by 1, the other half by 2. Label half of the children of the $0-$vertex by 0, the other half by 3. Group these vertices in perfect groups and label the descendants using the perfect group algorithm. It is easy to see that the complete $6-$ary tree labeled by the above algorithm is equitable at any height. $\square$

Lemma 11. *All complete* $7-$*ary trees are* $4-$*equitable.*

Proof. Given any complete $7-$ary tree, label the root vertex 0. Label the vertices in level 1 by $1, 2, 2, 3$ and $0, 1, 3$. We define the labeling of the descendants of the first group of 4 vertices as before: Label the children of the 1− vertex by 3. Label the children of the 3−vertex by 0. Label the children of one of the 2−vertices by 2 and label the children of the other 2−vertex by 1. Group these new vertices into perfect groups and label their descendants by using the perfect group algorithm repeatedly.

The labeling of the descendants of the remaining three vertices on level 1 is slightly more complicated. Label the children of the 3−vertex in level 1 with $0, 1, 2, 3$ and $0, 1, 3$. Label the children of the 0−vertex by $2, 2, 2, 3, 3, 3$ and a 0. Label the children of the 1−vertex by $0, 0, 0, 1, 1, 1$ and a 2.

We can group all but one of the level 2 vertices into perfect groups. At this point all edge labels are distributed evenly and 0 appears one more time as a vertex label than any other vertex label. Therefore, we can label the subsequent levels by using the perfect group algorithm and by treating the extra 0−vertex as a root vertex and repeating the previous labeling. The complete 7−ary tree labeled by the above algorithm is equitable at any height. □

Theorem 3. *All complete $n-$ary trees are $4-$equitable.*

Proof. Given any $n = 4k + l$ label the root vertex by 0. Then label $4(k - 1)$ vertices by equally distributing the 4 vertex labels. After grouping these vertices into perfect groups their descendants can be labeled by the perfect group algorithm. Label the remaining $4 + l$ vertices and their descendants based on the appropriate lemma above with the following modification:

- For $l = 1$, use $k - 1$ perfect groups for the descendants of the 3−vertex on level 1, the 3−vertex on level 2 and the 0−vertex on level 3.
- For $l = 0$ or $l = 2$, no modification is needed.
- For $l = 3$, use k perfect groups for the descendants of the 3−vetex on level 1 and then use the extra vertex labels $0, 1, 3$. For the other two level 1 vertices use the same pattern as in the case of 7−ary trees assigning labels 2 and 3 to the children of the 0−vertex and assigning labels 0 and 1 to the children of the 1−vertex with an extra 0− and 2−vertex, respectively.

Now we can proceed using the perfect labeling algorithm repeatedly and starting our whole process over regarding a 0−vertex on level 2 as a root vertex. □

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