2016

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Recommended Citation

Levin, D., Pudwell, L., Riehl, M., and Sandberg, A. (2016). Pattern avoidance in k-ary heaps. Australasian Journal of Combinatorics, 64(1), 120-139.

AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **64(1)** (2016), Pages 120–139

Pattern avoidance in k-ary heaps

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Abstract

In this paper, we consider pattern avoidance in k-ary heaps, where the permutation associated with the heap is found by recording the nodes as they are encountered in a breadth-first search. We enumerate heaps that avoid patterns of length 3 and collections of patterns of length 3, first with binary heaps and then more generally with k-ary heaps.

1 Introduction

In this paper, we continue a long line of research extending the notion of classical pattern avoidance in permutations to other structures. Given permutations $\pi = \pi_1\pi_2\cdots\pi_n$ and $\rho = \rho_1\rho_2\cdots\rho_m$ we say that π *contains* ρ as a pattern if there exist $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ such that $\pi_{i_a} < \pi_{i_b}$ if and only if $\rho_a < \rho_b$. In this case we say that $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_m}$ is *order-isomorphic* to ρ and that $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_m}$ *reduces to* ρ. If π does not contain ρ, then π is said to *avoid* ρ. This classical definition of pattern avoidance in permutations has seen broad application in areas ranging from the analysis of sorting algorithms to algebraic geometry. Analogues of pattern avoidance have been developed for a variety of combinatorial objects including Dyck paths [1], tableaux [5], set partitions [8], and more. In this paper, we are interested in analogues of pattern avoidance in ordered graph structures. In 2010, Rowland [7] introduced pattern avoidance in unlabeled binary trees. Essentially, tree T contains tree t if t is a contiguous, rooted, ordered subgraph of T. This definition of tree pattern was later generalized by the second author and coauthors to describe non-contiguous patterns in binary and k-ary trees [2, 3]. Here, tree T contains tree t as a non-contiguous pattern if there is a sequence of edge-contractions in T that produce t. More recently, in 2014, Yakoubov studied pattern avoidance in linear extensions of posets [9]. In particular, she considered specific classes of comb posets, which are essentially rooted, ordered, trees with vertices labeled in one of two canonical ways. For each type of comb poset and each canonical labeling, Yakoubov enumerated the linear extensions of the poset that produce permutations avoiding given patterns. In this paper, we extend the notion of pattern avoidance to a special kind of tree called a heap. Like work with comb posets, our heaps are labeled trees where each vertex has a label larger than its parent. Like work with unlabeled trees, we seek to determine the total number of heaps that avoid a given pattern, rather than work with linear extensions. Furthermore, our results for pattern avoidance in heaps, like pattern avoidance in unlabeled trees, can be translated into results for pattern-avoiding permutations where the permutations satisfy some additional structural restrictions. Although our work is motivated by previous work with pattern-avoiding trees and combs, our notion of pattern avoidance is distinct from the definitions used in previous work.

A *complete k-ary tree* is a tree where each node has k or fewer children, all levels except possibly the last are completely full (i.e. level i contains k^{i-1} vertices), and the last level has all its nodes to the left side (i.e. for any two vertices in the penultimate level, if the right-vertex has a positive out-degree, then the outdegree of the left vertex is k). A *k-ary heap* is a complete k-ary tree labeled with $\{1, \ldots, n\}$ such that every child has a larger label than its parent. We draw trees (respectively heaps) with the root at the bottom of the figure. An example of a 2-ary (i.e. binary) heap on 10 vertices in shown in Figure 1. Let $\mathcal{H}_n^k$ denote the set of k-ary n-vertex heaps. We see that the heap in Figure 1 is a member of $\mathcal{H}_{10}^2$. Notice that the heap in Figure 1 has 5 leaves. The number of leaves in a given n-vertex heap will be important throughout this paper, since many of our enumerations depend on the number of leaves. In general, an n-vertex binary heap has $\lceil \frac{n}{2} \rceil$ leaves. This is a straightforward computation: if i is the number of internal vertices and ℓ is the number of leaves, we have $i + \ell = n$. Each internal vertex, except possibly the rightmost vertex on the penultimate level, has outdegree 2 (in particular, when n is even the right-most internal vertex on the penultimate level has out-degree 1). We know the number of edges in an n-vertex heap is $n-1$, so, depending on the degree of the last internal vertex we have either $n - 1 = 2(i) = 2(n - \ell)$ or $n - 1 = 2(i - 1) + 1 = 2i - 1 = 2(n - \ell) - 1$. In either case, after solving for ℓ, we obtain $\lceil \frac{n}{2} \rceil$. A similar argument shows that an n-vertex k-ary heap has $\left\lceil \frac{(k-1)n-(k-2)}{k} \right\rceil$ leaves.

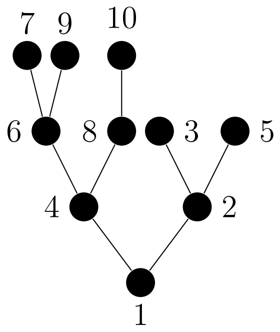


Figure 1: A binary heap with 10 vertices

Given a heap H, we associate a permutation π_H by recording the vertex labels as they are encountered in a breadth-first search. For example, if H is the heap in Figure 1, then $\pi_H = 142683579(10)$. We say that heap H contains (respectively avoids) ρ as a pattern if π_H contains (respectively avoids) ρ as a classical pattern, using the definition above. Let $\mathcal{H}_n^k(P)$ be the set of members of $\mathcal{H}_n^k$ that avoid all patterns in list P. While the heap in Figure 1 contains 123, 132, 213, 231, and 312, it is a member of $\mathcal{H}_{10}^2(321)$.

Throughout this paper, the main question is "how many elements are in $\mathcal{H}_n^k(P)$?" In general we fix $k = 2$ and a set of patterns P and then determine a formula for the sequence $\left\{\left|\mathcal{H}_n^k(P)\right|\right\}_{n\geq 1}$, with key results shown in Table 1. The third column of the table gives entries from the Online Encyclopedia of Integer Sequences [6]. The prevalence of sequences with low reference numbers indicates that our results for pattern-avoiding heaps have connections to other well-known combinatorial structures. Sequences A246747 and A246829, however, are new results particular to this study of heaps.

In Section 2 we consider heaps that avoid a single pattern of length 3. In Section 3 we consider heaps that avoid a pair of patterns of length 3, and in Section 4 we consider heaps avoiding three or more patterns of length 3. In Section 5 we generalize the results of the previous sections to k-ary heaps.

2 Heaps avoiding a pattern of length 3

Before we count pattern-avoiding heaps, it is instructive to enumerate *all* binary heaps. Let $a_n = |\mathcal{H}_n^2|$. It is immediate that $a_0 = 1$ and $a_1 = 1$. Now, for $n \geq 2$, notice that the root of a heap must have label 1, and the rest of the vertices in the

Patterns P	$\{\lvert\mathcal{H}_n^2(P)\rvert\}_{n>1}$	OEIS	Result
$\emptyset$	$1, 1, 2, 3, 8, 20, 80, 210, 896, \ldots$	A056971	Theorem 1
123	$1, 1, 1, 0, 0, 0, 0, 0, 0, \ldots$	A000004	Theorem 2
132	$1, 1, 1, 1, 1, 1, 1, 1, \ldots$	A000012	Theorem 3
213	$1, 1, 2, 2, 5, 5, 14, 14, 42, \ldots$	A208355	Theorem 4
231 312	$1, 1, 2, 3, 7, 14, 37, 80, 222, \ldots$	A246747	Theorem 5 Theorem 6
321	$1, 1, 2, 3, 7, 16, 45, 111, 318, \ldots$	A246829	OPEN
$\{213, 231\}$ $\{213, 312\}$	$1, 1, 2, 2, 4, 4, 8, 8, 16, \ldots$	A016116	Theorem 7 Theorem 8
$\{213, 321\}$	$1, 1, 2, 2, 4, 4, 7, 7, 11, \ldots$	A000124($\lceil \frac{n}{2} \rceil$)	Theorem 9
$\{231, 312\}$ $\{231, 321\}$ $\{312, 321\}$	$1, 1, 2, 3, 6, 11, 22, 42, 84, \ldots$	A002083	Theorem 10 Theorem 11 Theorem 12
$\{213, 231, 312\}$ $\{213, 231, 321\}$ $\{213, 312, 321\}$	$1, 1, 2, 2, 3, 3, 4, 4, 5, \ldots$	A008619	Theorem 13
$\{231, 312, 321\}$	$1, 1, 2, 3, 5, 8, 13, 21, 34 \ldots$	A000045	Theorem 14
$\{213, 231, 312, 321\}$	$1, 1, 2, 2, 2, 2, 2, 2, 2, \ldots$	A046698	Theorem 15

Table 1: Enumeration of pattern-avoiding binary heaps

heap partition into two smaller heaps. If we let

$$\begin{aligned} h &= \lfloor \log_2(n+1) \rfloor - 1, \\ b &= 2^h - 1, \\ r &= n - 1 - 2b, \\ r_1 &= r - \left\lfloor \frac{r}{2^h} \right\rfloor (r - 2^h), \\ r_2 &= r - r_1, \end{aligned}$$

then the left subheap has $b + r_1$ vertices and the right subheap has $b + r_2$ vertices. This is because h is the number of complete levels in the heap other than the root, b is the number of vertices on all complete levels in the left (respectively right) subheap, r is the number of vertices in the incomplete level at the top of the heap, and r_1 (respectively r_2) is the number of vertices in the incomplete level at the top of the left (respectively right) subheap. After choosing which of the numbers $\{2, \ldots, n\}$ will appear in the left subheap, we have the following recursive formula for a_n:

$$a_n = \binom{n-1}{b+r_1} a_{b+r_1} a_{b+r_2}.$$

Knuth provides an equivalent formula for the number of binary heaps on n vertices, computed as a product.

Theorem 1 ([4], Exercise 5.1.20)**.** *Given a binary heap on n vertices, let s_i be the size of the subtree whose root has label i, and let M_n be the multiset $\{s_1, s_2, \ldots, s_n\}$ of all these sizes. Then the number of heaps on n vertices is given by*

$$\frac{n!}{s_1 s_2 \cdots s_n} = \frac{n!}{\prod_{s \in M_n} s}.$$

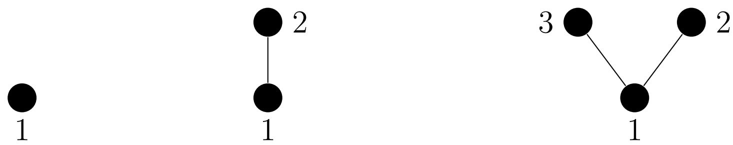
When we consider pattern-avoiding heaps the algebra is more straightforward, but the recursive process carries over. It turns out that heaps are more likely to contain lexicographically small patterns. This makes sense since the heap structure of H forces smaller digits to appear near the beginning of π_H. We consider all six patterns of length 3 in lexicographic order in the subsections below.

2.1 The patterns 123, 132, and 213

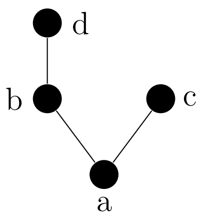
First, we consider the three lexicographically least patterns of length 3: 123, 132, and 213. Each of these allows for a simple and well-known enumeration sequence. The easiest pattern of length 3 to contain is 123.

Theorem 2. *Let $n \geq 1$. Then,* $|\mathcal{H}_n^2(123)| = \begin{cases} 1 & n \leq 3 \\ 0 & n \geq 4 \end{cases}.$

Proof. For $n \leq 3$, we show by exhaustion that there is exactly one heap avoiding 123. The appropriate heaps are shown below.



On the other hand, if $n \geq 4$, at the root our heap H contains the structure below where $a < b < d$. Since abd is a subsequence of π_H, H necessarily contains the pattern 123.



□

The analysis of 132-avoiding binary heaps is even simpler.

Theorem 3. *Let $n \geq 1$. Then, $|\mathcal{H}_n^2(132)| = 1$.*

Proof. Let $H \in \mathcal{H}_n^2(132)$. We know that the root must have label 1. Then, for π_H to avoid 132, all other labels must appear in increasing order. There is a unique heap where π_H is increasing for each $n \geq 1$. □

Finally, we consider heaps avoiding the pattern 213. This is our first non-trivial result, and it depends on the fact that an n-vertex binary heap has $\lceil \frac{n}{2} \rceil$ leaves.

Theorem 4. *Let $n \geq 1$ and let $C_n = \frac{\binom{2n}{n}}{n+1}$ be the n^{th} Catalan number. Then,*

$$\left|\mathcal{H}_n^2(213)\right| = C_{\lceil \frac{n}{2} \rceil}.$$

Proof. First, notice that in a heap avoiding 213, we cannot have any descents on internal nodes. If we had a descent beginning on an internal node, then the descent together with the child of the internal node would create an occurrence of a 213.

Further, we cannot have nonconsecutive labels on consecutive internal nodes. Assume we have nonconsecutive labels on a pair of consecutive internal nodes. Consider the first such occurrence. We have $a < b < c < d$ where a and c are labels on consecutive internal nodes, label b occurs later than a and c, and d is the label on a child of c. Notice that b must occur before d in the associated permutation, since each internal node after c is larger than c, precluding b from being its child. Then the heap has subword $acbd$, which contains the 213 pattern cbd.

Therefore, the labels on the internal nodes appear in consective increasing order. There are no restrictions on the labels for the leaves, except that the permutation must avoid 213. Such permutations are well-known to be counted by the Catalan numbers. Since there are $\lceil \frac{n}{2} \rceil$ leaves in an n-vertex binary heap, the number of heaps avoiding 213 is given by $C_{\lceil \frac{n}{2} \rceil}$. □

2.2 The patterns 231 and 312

The next two patterns, 231 and 312, turn out to have the same enumeration. First, we consider 231-avoiders.

Theorem 5. *Let $n \geq 1$ and let $C_n = \frac{\binom{2n}{n}}{n+1}$ be the nth Catalan number. Then, $|\mathcal{H}_1^2(213)| = 1$ and for $n \geq 2$,*

$$\left|\mathcal{H}_n^2(231)\right| = \sum_{i=0}^{\lceil \frac{n}{2} \rceil - 1} C_i \cdot \left|\mathcal{H}_{n-i-1}^2(231)\right|.$$

Proof. Consider $H \in \mathcal{H}_n^2(231)$. First, observe that the label n must appear on one of the $\lceil \frac{n}{2} \rceil$ leaves of H. That is $\pi_H = \pi_1\pi_2 \cdots \pi_{n-i-1} n \pi_{n-i+1} \cdots \pi_n$ where there are $0 \leq i \leq \lceil \frac{n}{2} \rceil - 1$ labels after n.

Further, since π_H avoids 231, we know the elements $\pi_1\pi_2 \cdots \pi_{n-i-1}$ are smaller than $\pi_{n-i+1} \cdots \pi_n$. The labels $\pi_{n-i+1} \cdots \pi_n$ must avoid 231, and since these labels all appear on leaves, there are C_i ways to arrange them so that they avoid 231. The elements $\pi_1\pi_2 \cdots \pi_{n-i-1}$, on the other hand, must form another 231-avoiding heap. There are $\left|\mathcal{H}_{n-i-1}^2(231)\right|$ ways to build a 231-avoiding heap with $n - i - 1$ vertices, so summing over all possible values for i yields the result. □

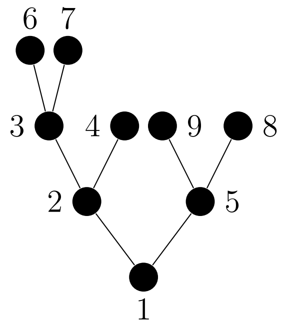
It turns out that $|\mathcal{H}_n^2(312)|$ satisfies the same recurrence. However, we prove this with a bijection rather than via direct enumeration.

Theorem 6. *Let $n \geq 1$. Then,*

$$\left|\mathcal{H}_n^2(231)\right| = \left|\mathcal{H}_n^2(312)\right|.$$

Proof. We define a bijection $\phi : \mathcal{H}_n^2(231) \to \mathcal{H}_n^2(312)$. First, if $H \in \mathcal{H}_n^2(231) \cap \mathcal{H}_n^2(312)$, $\phi(H) = H$.

Now, let $H \in \mathcal{H}_n^2(231) \setminus \mathcal{H}_n^2(312)$. Let M be the set of left-to-right maxima of π_H. For example, in the heap shown below, $\pi_H = 125349867$ and $M = \{1, 2, 5, 9\}$.

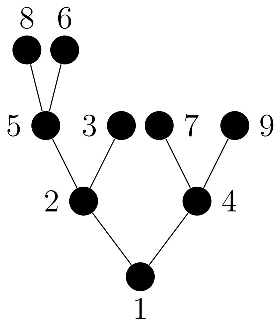


Let $p = |M|$ and order the elements of M such that $m_1 < m_2 < \cdots < m_p$. Let M_i be the elements of π_H after and including m_i but before m_{i+1} for $1 \leq i \leq p - 1$. Let M_p be the subpermutation beginning with m_p and continuing to the end of π_H. We make the following observations.

- The elements of M appear in π_H in increasing order. This is by definition of left-to-right maxima.
- If an element $m \in M$ appears in a 312 pattern, it must play the role of '3'. If, on the contrary, m plays the role of a '1' or '2' then there is an element to the left of m and larger than m that plays the role of '3' instead. This contradicts the fact that m is a left-to-right maximum.
- All elements of M_{i+1} are larger than all elements of M_i. If not, π_H contains a 231 pattern where m_{i+1} plays the role of '3'.
- Two elements $m^*, m' \in M_i$ may not appear with m^* above m' in H. If there existed two such elements, then there would be another element of M_i that appears above m_i, which is the largest element of M_i. This contradicts the definition of heap.

These observations imply that each M_i consists of a consecutive set of integers on consecutive vertices of H and that we may permute elements within an individual M_i without destroying heap structure. We seek to transform a 231-avoiding heap with j copies of 312 into a 312-avoiding heap with j copies of 231. To this end, for each M_i let M_i^* be the set of elements that play the role of '2' in a 312 pattern where m_i plays the role of '3'. Notice that M_i^* is exactly the set of elements of M_i appearing after $\min(M_i)$. Move all these elements before m_i to create copies of 231. Do this on each M_i to obtain $\phi(H)$.

For example, the heap above with $\pi_H = 125349867$ would map to $\phi(\pi_H) = 124537986$, which corresponds to the heap shown below.



To produce $\phi^{-1} : \mathcal{H}_n^2(312) \to \mathcal{H}_n^2(231)$, again if $H \in \mathcal{H}_n^2(231) \cap \mathcal{H}_n^2(312)$, $\phi^{-1}(H) = H$.

Now, suppose $H \in \mathcal{H}_n^2(312) \setminus \mathcal{H}_n^2(231)$ and let M be the set of right-to-left minima of π_H. Let $p = |M|$ and order the elements of M such that $m_1 < m_2 < \cdots < m_p$. Let M_i be the elements of π_H before and including m_{i+1} but after m_i for $2 \leq i \leq p$. We make the following observations.

- The elements of M appear in π_H in increasing order. This is by definition of right-to-left minima.
- If an element $m \in M$ appears in a 231 pattern, it must play the role of '1'. If, on the contrary, m plays the role of a '2' or '3' then there is an element to the right of m and smaller than m that plays the role of '1' instead. This contradicts the fact that m is a right-to-left minimum.
- All elements of M_{i+1} are larger than all elements of M_i. If not, π_H contains a 312 pattern where m_{i+1} plays the role of '1'.
- Two elements $m^*, m' \in M_i$ may not appear with m^* above m' in H. If there existed two such elements, then there would be another element of M_i that appears below m_{i+1}, which is the smallest element of M_i. This contradicts the definition of heap.

These observations imply that each M_i consists of a consecutive set of integers on consecutive vertices of H and that we may permute elements within an individual M_i without destroying heap structure. We seek to transform a 312-avoiding heap with j copies of 231 into a 231-avoiding heap with j copies of 312. To this end, for each M_i let M_i^* be the set of elements that play the role of '2' in a 231 pattern where m_{i+1} plays the role of '1'. Notice that M_i^* is exactly the set of elements of M_i appearing before $\max(M_i)$. Move all these elements after m_{i+1} to create copies of 312. Do this on each M_i to obtain $\phi^{-1}(H)$. □

n	$\lvert\mathcal{H}_n^2(321)\rvert$	n	$\lvert\mathcal{H}_n^2(321)\rvert$	n	$\lvert\mathcal{H}_n^2(321)\rvert$
1	1	11	2686	21	395303480
2	1	12	8033	22	1379160685
3	2	13	25470	23	4859274472
4	3	14	80480	24	17195407935
5	7	15	263977	25	61310096228
6	16	16	862865	26	219520467207
7	45	17	2891344	27	790749207801
8	111	18	9706757	28	2859542098634
9	318	19	33178076	29	10391610220375
10	881	20	113784968	30	37897965144166
				31	138779392289785

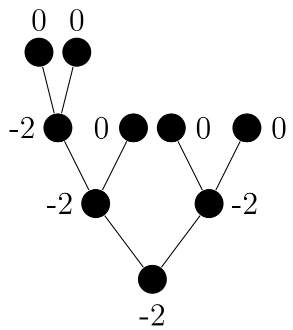
Table 2: The number of 321-avoiding binary heaps for $n \leq 31$

2.3 The pattern 321

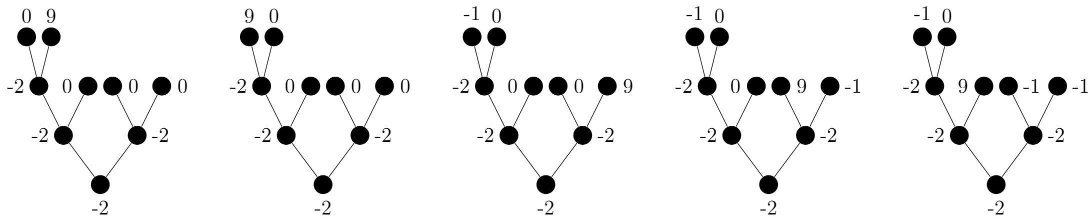
Enumeration of 321-avoiding heaps remains open. We have computed $\{|\mathcal{H}_n^2(321)|\}_{n=1}^{31}$ as shown in Table 2 using the recursive technique described below.

Notice that in any n-vertex heap, the label n must appear on a leaf. Further, in a 321-avoiding heap, all labels after the n must appear in increasing order. Therefore, if n appears in position i in π_H, $n-1$ may not appear in positions $i+1, \ldots, n-1$, but it may appear before the label n or in position n.

Our recursive technique begins with a binary tree with n vertices but with labels chosen from $\{-2, -1, 0\}$ as shown below. Leaves initially have label 0, which indicates that n may be placed on that vertex without violating the creation of a 321-avoiding heap. The internal vertices initially have label -2, which indicates that that vertex has children who have not yet received a label. A label of -1 will be used in subsequent iterations to mark a leaf with a labeled vertex to its left and unlabeled vertices to its right.



The children of this tree consist of all choices where the largest unused label from $\{1, \ldots, n\}$ has been placed on a 0 vertex, and the other vertices' labels are updated to the appropriate label from $\{-2, -1, 0\}$. For example, the tree above has 5 leaves and thus 5 children. They are shown below.



By iterating this process, n times, we eventually obtain each heap in $\mathcal{H}_n^2(321)$ exactly once. Often, we iterate and produce a heap where the final i vertices have labels and the first $n-i$ vertices do not. In this case, we can recursively compute $\left|\mathcal{H}_{n-i}^2(321)\right|$ to determine the number of children more quickly. While brute force techniques become time-consuming for heaps with 10 or more vertices; the labeling conventions above have allowed us to compute the number of 321-avoiding heaps with as many as 31 vertices.

Although we do have have a closed formula for $|\mathcal{H}_n^2(321)|$, we may place some bounds on $|\mathcal{H}_n^2(321)|$.

Lemma 1. *For $n \geq 9$, $2^{n-1} < |\mathcal{H}_n^2(321)| < 4^n$.*

Proof. For the upper bound, notice that $\mathcal{H}_n^2(321) \subseteq \mathcal{S}_n(321)$, and it is well-known that $|\mathcal{S}_n(321)| = C_n < 4^n$, where $C_n = \dfrac{\binom{2n}{n}}{n+1}$ is the nth Catalan number.

On, the other hand, for the lower bound we give a constructive argument. When $n = 9$, we have $2^8 = 256 < 318 = |\mathcal{H}_9^2(321)|$. Now, for $n \geq 9$, we need only show that for $H \in \mathcal{H}_n^2(321)$, there exist at least two heaps $H^*, H' \in \mathcal{H}_{n+1}^2(321)$ that can be generated from H. First, given $H \in \mathcal{H}_n^2(321)$, let H^* be the heap obtained by adding $(n+1)$ as a new last leaf vertex. Second, to generate $H' \in \mathcal{H}_{n+1}^2(321)$, consider separate cases for if $n+1$ is even or odd. If $(n+1)$ is odd, then the last two leaves of H share the same parent, so we may also add $(n+1)$ as the penultimate leaf without creating a 321-pattern. If $(n+1)$ is even, then consider three subcases to create H' from H. (i) If the first leaf i is smaller than the last leaf j, then insert $n+1$ as the penultimate leaf, and i will become j's parent. (ii) If the first leaf i is larger than the last leaf j and $i \neq n$, then put $(n+1)$ in n's place and put n as the new last leaf. (iii) If the first leaf is n, then the last leaf must be $(n-1)$. Exchange the locations and n and $(n-1)$. Now the last 2 leaves are a and n for some $a < n$. Change a to $(n+1)$, change n to a, and add a new leaf above $(n-1)$ with label n.

We have generated

$$\bigcup_{H \in \mathcal{H}_n^2(321)} \{H^*, H'\} \subseteq \mathcal{H}_{n+1}^2(321)$$

where $\left|\bigcup_{H \in \mathcal{H}_n^2(321)} \{H^*, H'\}\right| = 2\,|\mathcal{H}_n^2(321)|$, which shows that $2^{n-1} < |\mathcal{H}_n^2(321)|$ for $n \geq 9$. □

It is clear, then, that $|\mathcal{H}_n^2(321)|$ grows exponentially. It remains to determine $c \in (2,4)$ such that $|\mathcal{H}_n^2(321)| \sim c^n$. Based on numerical data, we conjecture that $c > 3.66$.

To get a better understanding of $|\mathcal{H}_n^2(321)|$, we also consider the sequence $\frac{|\mathcal{H}_{n+1}^2(321)|}{|\mathcal{H}_n^2(321)|}$. A sequence $a_1, a_2, \ldots$ is said to be *log-convex* if $a_i^2 \leq a_{i-1}a_{i+1}$ for $i \geq 2$. In other words, a sequence is log-convex if the sequence of ratios of consecutive terms is weakly increasing. When we compute these ratios with the terms in Table 2, the only times when $\frac{|\mathcal{H}_{n+1}^2(321)|}{|\mathcal{H}_n^2(321)|} > \frac{|\mathcal{H}_{n+2}^2(321)|}{|\mathcal{H}_{n+1}^2(321)|}$ are when $n \in \{2, 4, 6, 8, 10, 12, 14\}$. Because we are concerned with binary heaps, it also makes sense to consider the sequences $\left\{\left|\mathcal{H}_{2i+1}^2(321)\right|\right\}_{i\geq 0}$ and $\{|\mathcal{H}_{2i}^2(321)|\}_{i\geq 1}$ separately. When we consider the ratio of consecutive terms in each of these sequences, the ratios are always increasing for the data in Table 2.

To that end, based on the 31 terms in Table 2, we make the following conjectures.

Conjecture 1. *The following are true for* $|\mathcal{H}_n^2(321)|$*:*

1. $|\mathcal{H}_n^2(321)| \sim c^n$ *for some constant* $c \in (3.66, 4)$.
2. $\{|\mathcal{H}_n^2(321)|\}_{n\geq 15}$ *is log-convex.*
3. $\left\{\left|\mathcal{H}_{2i+1}^2(321)\right|\right\}_{i\geq 0}$ *is log-convex.*
4. $\{|\mathcal{H}_{2i}^2(321)|\}_{i\geq 1}$ *is log-convex.*

3 Heaps avoiding a pair of patterns of length 3

Next, we study pairs of patterns of length 3. While there are 15 such pairs of patterns, we focus on the $\binom{4}{2} = 6$ pairs of patterns $\{\sigma, \tau\}$ where both $|\mathcal{H}_n^2(\sigma)|$ and $|\mathcal{H}_n^2(\tau)|$ are non-trivial.

3.1 Heaps avoiding $\{213, 231\}$ or $\{213, 312\}$

Just as $|\mathcal{H}_n^2(231)| = |\mathcal{H}_n^2(312)|$, these enumerations still agree when we add the extra restriction of avoiding 213. First we consider heaps avoiding both 213 and 231.

Theorem 7. *Let* $n \geq 1$. *Then,*

$$\left|\mathcal{H}_n^2(213, 231)\right| = 2^{\left\lceil \frac{n}{2} \right\rceil - 1}.$$

Proof. As we know from the proof of Theorem 5, to create a heap avoiding 231, we must have n on a leaf and all labels before n must be less than all labels after n. Additionally, to avoid 213, all labels before n must be in increasing order. Combining these two facts, we see that all interior nodes must be labeled consecutively starting with 1. It remains to label the $\left\lceil \frac{n}{2} \right\rceil$ leaves of the heap with the labels $\left\{n - \left\lceil \frac{n}{2} \right\rceil + 1, \ldots, n\right\}$ in a way that avoids 213 and 231. It is well known that the number of permutations of length i avoiding 213 and 231 is given by 2^{i-1} (filling in the permutation from left to right, the next element must always be either the smallest or the largest of the remaining elements), so replacing i with $\left\lceil \frac{n}{2} \right\rceil$ gives the theorem. □

It turns out that heaps avoiding both 213 and 312 have the same enumeration.

Theorem 8. *Let $n \geq 1$. Then,*

$$\left|\mathcal{H}_n^2(213, 312)\right| = 2^{\lceil \frac{n}{2} \rceil - 1}.$$

Proof. From the proof of Theorem 4 we know that the internal nodes of $H \in \mathcal{H}_n^2(213, 312)$ must be consecutive integers. We need only consider the number of ways to form a $\{213, 321\}$-avoiding permutation with the $\lceil \frac{n}{2} \rceil$ largest labels on on the leaves.

Given a permutation that avoids both 213 and 312 we see that after the first descent all numbers must appear in decreasing order, lest we create a forbidden pattern. Therefore, if $\ell = \lceil \frac{n}{2} \rceil$ is the number of leaves of H, we choose $0 \leq i \leq \ell - 1$ of the $\ell - 1$ labels in $\{n - \ell + 1, \ldots, n - 1\}$ to appear in increasing order before n, and then the remaining labels appear in decreasing order after n. Summing over all possible values of i yields

$$\sum_{i=0}^{\ell-1} \binom{\ell - 1}{i} = 2^{\ell-1}$$

possible heaps, and replacing ℓ with $\lceil \frac{n}{2} \rceil$ gives the theorem. □

3.2 Heaps avoiding $\{213, 321\}$

It turns out that heaps avoiding 213 and 321 are enumerated by another simple combinatorial formula.

Theorem 9. *Let $n \geq 1$. Then,*

$$\left|\mathcal{H}_n^2(213, 321)\right| = 1 + \binom{\lceil \frac{n}{2} \rceil}{2}.$$

Proof. Suppose $H \in \mathcal{H}_n^2(213, 321)$. Then H has $\ell = \lceil \frac{n}{2} \rceil$ leaves. We observe the following:

- Because H is a heap, n appears on a leaf.
- Because H avoids 321, all labels after n are increasing.
- Because H avoids 213, all labels before n are increasing. Furthermore, all labels after n form a consecutive set of integers. Otherwise, there exists b before n and labels a and c after n where a appears before c, $a < b < c$, and bac forms a 213 pattern in π_H.
- The $n - \ell$ internal vertices must have labels $1, 2, \ldots, n - \ell$. Otherwise, consider the label i of the last internal vertex of H. If $i > n - \ell$, there is some label less than i that is not used on an internal vertex. Let j be the largest such label. By the observations above, the last $\ell + 1$ digits of π_H must be $i(i+1)(i+2)\cdots n$ followed by a consecutive increasing run of integers ending with j. Since i is the label of the *last* internal vertex, and j is the label of the last leaf, j must be i's child in H. But $j < i$, which contradicts the definition of heap.

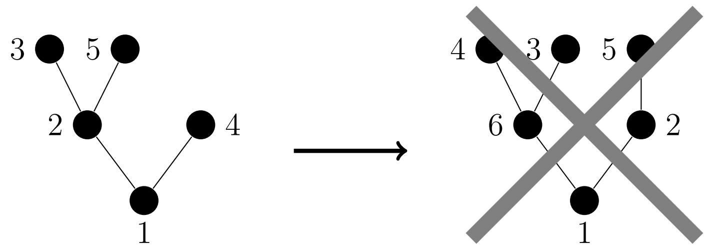
Therefore, with $H \in \mathcal{H}_n^2(213, 321)$, the labels of the internal vertices are already determined, and it only remains to label the leaves with $\{n-\ell+1, \ldots, n\}$. If n appears on the last leaf, then π_H must be the identity permutation. Otherwise, if there are $i \geq 1$ leaves after n, there are $\ell - i$ ways to choose which consecutive labels appear after n. Summing over all possible values of i, we obtain $1 + \sum_{i=1}^{\ell-1}(\ell - i) = 1 + \binom{\ell}{2}$ possible heaps avoiding both 213 and 321. Replacing ℓ with $\lceil \frac{n}{2} \rceil$ gives the theorem. □

3.3 Heaps avoiding $\{231, 312\}$, $\{231, 321\}$, or $\{312, 321\}$

Our remaining three pairs of patterns all yield the same enumeration. It turns out that with an offset of one term, the sequence $\{|\mathcal{H}_n^2|\}_{n\geq 1}$ is given by the Narayana-Zidek-Capell numbers (OEIS A002083), which have appeared in problems involving single-elimination tournaments, lattice paths, and trees. Our pattern-avoiding heaps give yet another appearance of this recursive sequence.

Theorem 10. *Let $a_n = |\mathcal{H}_n^2(231, 312)|$. Then, $a_1 = 1$, $a_2 = 1$, and for $n \geq 3$, $a_n = 2a_{n-1}$ when n is odd, and $a_n = 2a_{n-1} - a_{\frac{n-2}{2}}$ when n is even.*

Proof. To count $\{231, 312\}$-avoiders, we make an insertion argument. We begin with a heap on $n-1$ nodes that avoids $\{231, 312\}$, insert n at some point in the permutation and move all the node labels forward after it, but leave it an increasing tree. Below we show a heap on 5 vertices where, after inserting the label 6, we obtain a non-increasing tree.

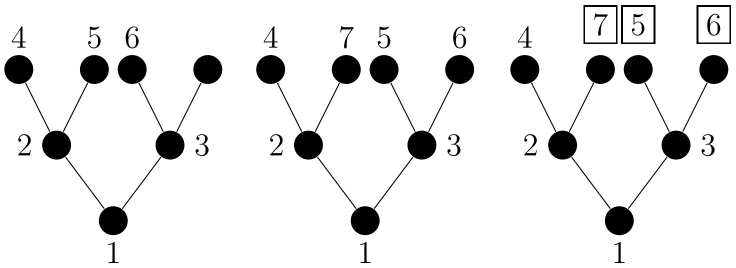


We make two key observations about insertion:

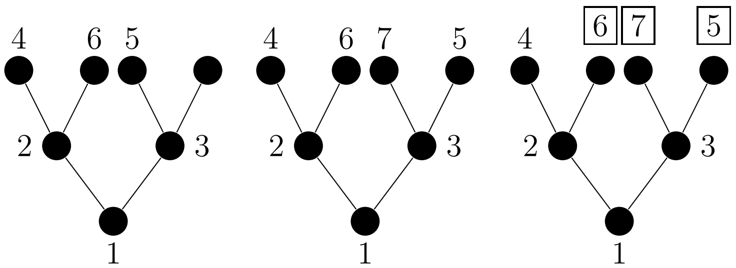
1. The vertex labeled $n-1$ is always a leaf before insertion. After insertion, the vertex labeled n is always a leaf.

2. In order for the associated permutation to avoid 231 and 312, n must be inserted directly before $n-1$ or at the end of the heap.

The first observation follows directly from properties of heaps, while the second observation takes more thought. If we insert n directly before $n-1$, we have not created any occurrences of 231 nor 312, or there already would have been one present before insertion. If we insert n at the end, we have not created any occurrences of 231 nor 312. We cannot insert n anywhere further before $n-1$, or we will create a 312 pattern as shown below.



Further, we cannot insert n after $n-1$ but not at the end or we have created a 231 pattern, as shown below.



Now, we consider two cases.

Case 1: n is odd.

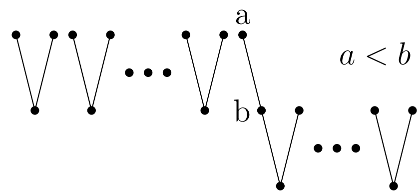
Since n is odd, the new leaf added to the heap is the sibling of a current leaf. Thus all nodes that were internal nodes before insertion stay internal nodes and their labels do not change, and all leaves stay leaves, plus the new sibling of the final leaf is added. Since $n-1$ was on a leaf before insertion, $n-1$ and n are both on leaves after insertion. Some of the leaf labels may have shifted to be the children of a new root, and we should carefully examine why our tree remains increasing after this shift. The only way we could have made a non-increasing tree would be by shifting a right leaf label before insertion to be a left leaf after insertion, and that its label is now smaller than its new parent. This cannot occur, for if such a situation existed, then before insertion, the offending parent node, the node that was labeled $n-1$ and the node that will be shifted form a 231. Therefore our insertion process cannot have created a tree that is not increasing. Then we have exactly two choices for creating a heap of size n from each heap of size $n-1$, so $a_n = 2a_{n-1}$.

Case 2: n is even.

In this case, the new leaf added to the heap is the child of a node which was a leaf before insertion. As such this new parent node might have a large label since it was a leaf before insertion. So unlike Case 1 above, there is the possibility that inserting n immediately before $n-1$ might create a tree that is no longer increasing. We examine each insertion (n at the end of the permutation, n immediately before $n-1$ but $n-1$ was not on the first leaf, and n immediately before $n-1$ when $n-1$ was on the first leaf) to see which options cause forbidden labelings for our trees.

If we insert n at the end, it becomes the child of our former leaf, but the tree is still increasing, no matter what the label of our new parent node was.

If we insert n immediately before $n-1$, but not on the first leaf, we must consider if we could have the situation pictured below, where because a is less than b, we no longer have a heap. However, this situation is impossible, because we must already have had a 231 pattern in our heap before insertion, namely $b(n-1)a$. Thus all instances for this insertion form legal heaps.



If we insert n immediately before $n-1$ on the first leaf, then n becomes the parent of a node with a (guaranteed) smaller label. We do not want to count these possibilities, even though they may still avoid 231 and 312, because they are not heaps. Since $n-1$ was on the first leaf before insertion, (and after insertion, as a matter of fact), all the labels after $n-1$ are in decreasing order. The subtree obtained by removing all leaves needs to avoid 231 and 312, and there are $\frac{n-2}{2}$ nodes on that subtree. Thus there are $a_{\frac{n-2}{2}}$ ways to have inserted n immediately before $n-1$ on the first leaf and created trees that are no longer increasing.

Summing the possibilities from Case 1 and Case 2, of the $2a_{n-1}$ ways to insert n while still avoiding $\{231, 312\}$, $a_{\frac{n-2}{2}}$ create trees that are not increasing, so there are $a_n = 2a_{n-1} - a_{\frac{n-2}{2}}$ ways to create a heap on n vertices that avoids $\{231, 312\}$. Thus we have the same recurrence relation and initial condition as the Narayana-Zidek-Capell numbers offset by a single starting term, proving our theorem. ☐

It turns out that $\{231, 321\}$-avoiding heaps and $\{312, 321\}$-avoiding heaps have the same enumeration.

Theorem 11. *Let* $n \geq 1$. *Then,*

$$\left|\mathcal{H}_n^2(231, 312)\right| = \left|\mathcal{H}_n^2(231, 321)\right|.$$

Proof. We wish to define a bijection $\phi : \mathcal{H}_n^2(231, 312) \to \mathcal{H}_n^2(231, 321)$.

Consider $H \in \mathcal{H}_n^2(231, 312)$ and let M be the set of left-to-right maxima of π_H. Let $p = |M|$ and label the elements of M such that $m_1 < m_2 < \cdots < m_p$. We observe the following:

- All elements after m_i and less than m_i appear in decreasing order because π_H avoids 312.
- All elements between m_i and m_{i+1} are less than the set of elements after m_{i+1} because π_H avoids 231.

Therefore, to compute $\phi(\pi_H)$, for $1 \leq i \leq p-1$ arrange the elements between m_i and m_{i+1} in increasing order. Then put the elements after m_p in increasing order. ϕ is clearly invertible. ☐

For our last pair of patterns, we make another insertion argument.

Theorem 12. *Let $n \geq 1$. Then,*

$$\left|\mathcal{H}_n^2(231, 312)\right| = \left|\mathcal{H}_n^2(312, 321)\right|.$$

Proof. Consider $H \in \mathcal{H}_{n-1}^2(312, 321)$. To avoid 312, every label after $n-1$ in H is in decreasing order. Similarly, to avoid 321, every label after $n-1$ in H must be in increasing order. Thus H has $n-1$ on the last or second-to-last leaf.

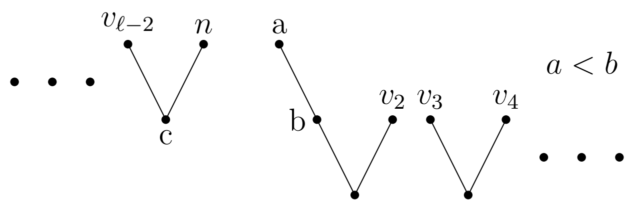
We will now insert n to create elements of $\mathcal{H}_n^2(312, 321)$. We must insert n at the end of the heap, or at the penultimate position. In either case, the heap still avoids 321 and 312. But does the insertion leave us with a legal heap?

Case 1: n is odd.

In this case, the leaf added to insert n has a sibling, so whether we insert at the ultimate or penultimate position, we still have a legal heap, because no parent-child relationships have changed except that the parent of the last leaf has a new child labeled n. Thus each element of $\mathcal{H}_{n-1}^2(312, 321)$ yields 2 elements in $\mathcal{H}_n^2(312, 321)$, so $a_n = 2a_{n-1}$ when n is odd.

Case 2: n is even.

In this case, when we add a new leaf to our heap, a node that was a leaf becomes an interior node. When we insert n at the end, we create a legal heap. However, when we insert n before the last element, we might end up with a leaf with a label smaller than that of its parent, as pictured below.



How many ways can such a situation happen? Let $\ell = \frac{n}{2}$, the number of leaves on the heap. Label the vertices between a and b with v_2 through $v_{\ell-1}$. Then we know that $v_{\ell-1} = n$ and $b > a$. Additionally, we know that for all $1 < i < \ell - 1$, we know $v_i > b$, otherwise bv_ia forms a 321 pattern.

Next, notice that all interior node labels are less than b, otherwise combined with b and a we would form a 321 pattern. But in fact, significantly more is true, because all interior nodes labels are less than a, though this is a bit more subtle! The penultimate interior node (labeled c in the diagram above) is less than a since a was its child before insertion. Then all nodes before c are also less than a, otherwise we would form a 312 pattern with c and a. The v_i's are in increasing order. If they were not, consider the first descent v_iv_{i+1}, and see that $v_iv_{i+1}a$ forms a 321 pattern.

Now, $a = \frac{n}{2}$, $b = a + 1$, $v_2 = b + 1$, and for all $2 \leq i \leq \ell - 2$, $v_i = v_{i-1} + 1$. All leaf labels are completely determined in this situation, so we need only ensure that we had the subheaps obtained by removing all the leaves (including the former leaf b) avoiding 312 and 321. There are $a_{\frac{n-2}{2}}$ such heaps. So when n is even, $a_n = 2a_{n-1} - a_{\frac{n-2}{2}}$.

□

4 Heaps avoiding three or four patterns of length 3

There are only five nontrivial cases to examine when we avoid a triple or quadruple of patterns of length 3:

$$\{213, 231, 312\}, \{213, 231, 321\}, \{213, 312, 321\}, \{231, 312, 321\}, \text{ and } \{213, 231, 312, 321\}.$$

It turns out three of the four triples of patterns yield the same enumeration.

Theorem 13. *For* $n \geq 1$,

$$\left|\mathcal{H}_n^2(213, 231, 312)\right| = \left|\mathcal{H}_n^2(213, 231, 321)\right| = \left|\mathcal{H}_n^2(213, 312, 321)\right| = \left\lceil \frac{n}{2} \right\rceil.$$

Proof. We consider each pattern set in turn. If π_H avoids 213, 231, and 312 then $\pi_H = 12 \cdots in(n-1)(n-2) \cdots (i+1)$ for some i. Therefore, choosing the location of n uniquely determines the permutation. Since n may only appear on one of the $\lceil \frac{n}{2} \rceil$ leaves of H, there are $\lceil \frac{n}{2} \rceil$ heaps avoiding this pattern set.

Next, if π_H avoids 213, 231, and 321, then $\pi_H = 12 \cdots in(i+1)(i+2) \cdots (n-1)$. Again, choosing the location of n uniquely determines the permutation. Placing n on one of the $\lceil \frac{n}{2} \rceil$ leaves of H gives the desired enumeration.

Finally, if π_H avoids 213, 312, and 321 we know that either $\pi_H = 12 \cdots n$, or there exists some $j < n$ such that $\pi_H = 12 \cdots (j-1)(j+1) \cdots nj$. There are $\lceil \frac{n}{2} \rceil - 1$ values that may play the role of j without placing a larger value on j's parent, so together with the heap corresponding to the identity permutation there are again $\left(\lceil \frac{n}{2} \rceil - 1\right) + 1 = \lceil \frac{n}{2} \rceil$ possible heaps. □

The final triple of patterns, $\{231, 312, 321\}$ gives a more interesting enumeration.

Theorem 14. *Let* $n \geq 1$. *Then,*

$$\left|\mathcal{H}_n^2(231, 312, 321)\right| = F_n,$$

where F_n *is the* n^{th} *Fibonacci number, with* $F_1 = 1, F_2 = 1$.

Proof. Each heap avoiding 312 and 321 must have n as the ultimate or penultimate label. Furthermore, since the heap avoids 231, every label before n is smaller than every label after n, so the heap ends in n or $n(n-1)$. This gives us a recursive manner with which to build larger heaps. Take a heap of size $n-1$ which avoids $\{231, 312, 321\}$, add a new leaf, and label it n. Or, take a heap of size $n-2$ which avoids $\{231, 312, 321\}$, add two new leaves, and label them n and $n-1$. Our sequence follows the Fibonacci recurrence, and it is easy to check the base cases hold. □

Finally, we count heaps avoiding the 4-tuple $\{213, 231, 312, 321\}$.

Theorem 15.

$$\left|\mathcal{H}_n^2(213, 231, 312, 321)\right| = \begin{cases} 1 & n = 1 \text{ or } n = 2 \\ 2 & n \geq 3. \end{cases}$$

Proof. Suppose π_H avoids the four given patterns. Since π_H avoids 312 and 321, n must either be the last or the penultimate digit of π_H. If n is last, then since π_H avoids 213, everything before n is increasing and we have the identity permutation. If n is the penultimate digit, because π_H avoids 231, the last digit must be $n-1$, and because π_H avoids 213, all digits before n must be increasing so we have the permutation $12\cdots(n-2)n(n-1)$. Both of these permutations can be written on any heap with $n \geq 3$ vertices, so there are exactly 2 pattern-avoiding heaps for $n \geq 3$. □

5 Generalization to k-ary heaps

At this point we have enumerated binary heaps avoiding any set of patterns of length 3 other than the singleton pattern 321. Enumeration of 321-avoiders was already challenging for binary heaps and their enumeration remains an open problem for k-ary heaps as well. However, the rest of our results generalize nicely to k-ary heaps. We know that the number of leaves in a k-ary heap with n vertices is given by $\ell = \left\lceil \frac{(k-1)n-(k-2)}{k} \right\rceil$. Since Theorems 4, 5, 6, 7, 8, 9, and 13 depend on the number of leaves in the heap rather than the number of vertices, replacing $\lceil \frac{n}{2} \rceil$ with $\ell = \left\lceil \frac{(k-1)n-(k-2)}{k} \right\rceil$ gives the corresponding generalized results seen in Table 3. Theorems 2, 3, 14, and 15 generalize similarly. It remains to find a formula for $\left|\mathcal{H}_n^k(231,312)\right| = \left|\mathcal{H}_n^k(231,321)\right| = \left|\mathcal{H}_n^k(312,321)\right|$, which we present in Theorem 16.

Lemma 2. *Let $n > 2$. In a k-ary heap, if $k \mid n-2$, there is a leaf with no siblings, in particular it is the last leaf. Otherwise if $k \nmid n-2$, every leaf has a sibling.*

Theorem 16. *Let $a_n = \left|\mathcal{H}_n^k(231,312)\right|$. Then*

$$a_n = \begin{cases} 1 & n \leq 2 \\ 2a_{n-1} & k \nmid n-2 \\ 2a_{n-1} - a_{\frac{n-2}{k}} & k \mid n-2. \end{cases}$$

Proof. In an effort to be concise, we note that the argument used in Theorem 10 is exactly the argument required here. In Theorem 10, the differentiation of the cases was only based on whether the new leaf added was added to a node that was already a parent node, or whether it was added as the first child to a node that was formerly a leaf. By Lemma 2, we see that those two cases are covered by $k \mid n-2$ and $k \nmid n-2$ when we generalize to k-ary trees, and thus the same recurrence holds.

□

Many of the results in Table 3 rely on the number of leaves in the heap rather than the number of vertices. It turns out that this is merely an artifact of avoiding the pattern 213. We showed in Theorem 4 that if h avoids 213 then the i internal vertices of h must have the labels $1, 2, \ldots, i$ in increasing order. This phenomenon does not change when h avoids a set of patterns including 213. However, in general, avoiding sets of patterns of length $m \geq 4$ does not produce sequences that depend on the number of leaves. On the other hand, we note that if $\rho \in \mathcal{S}_m$ such that $\rho_1 \neq 1$, then heap

Patterns P	$\lvert\mathcal{H}_n^k(P)\rvert$ (where $\ell = \left\lceil \frac{(k-1)n-(k-2)}{k} \right\rceil$)
123	$\begin{cases} 1 & n \leq k+1 \\ 0 & n \geq k+2 \end{cases}$
132	1
213	C_ℓ
231 312	$\begin{cases} 1 & n = 1 \\ \sum_{i=0}^{\ell-1} C_i \cdot \lvert\mathcal{H}_{n-i-1}^k(231)\rvert & n \geq 2 \end{cases}$
321	OPEN
$\{213, 231\}$ $\{213, 312\}$	$2^{\ell-1}$
$\{213, 321\}$	$\binom{\ell}{2} + 1$
$\{231, 312\}$ $\{231, 321\}$ $\{312, 321\}$	$\begin{cases} 1 & n \leq 2 \\ 2a_{n-1} & k \nmid n-2 \\ 2a_{n-1} - a_{\frac{n-2}{k}} & k \mid n-2. \end{cases}$
$\{213, 231, 312\}$ $\{213, 231, 321\}$ $\{213, 312, 321\}$	ℓ
$\{231, 312, 321\}$	F_n
$\{213, 231, 312, 321\}$	$\begin{cases} 1 & n \leq 2 \\ 2 & n \geq 3 \end{cases}$

Table 3: Enumeration of pattern-avoiding k-ary heaps

H avoids ρ if and only if H avoids $1 \oplus \rho$, so $|\mathcal{H}_n^2(213)| = |\mathcal{H}_n^2(1 \oplus 213)| = |\mathcal{H}_n^2(1324)|$; that is, 1324 is the unique pattern of length 4 that produces an enumeration sequence depending on the number of leaves.

Acknowledgement

The authors are grateful to two anonymous referees for a number of insightful comments that improved the presentation of this paper.

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(Received 12 Sep 2014; revised 8 Mar 2015)