Fantastic Patterns and Where Not to Find Them

Michael Bukata, Ryan Kulwicki, Nicholas Lewandowski, Jake Roth, Teresa Wheeland
Advisor: Dr. Lara Pudwell

Abstract

Interesting patterns are everywhere we look, but what happens when we try to avoid patterns? A permutation is a list of numbers in a specific order. When we avoid a pattern, we try not to order those numbers in certain ways. For example the permutation 45312 avoids the 123 pattern because no three elements in the permutation are in an increasing order. In our work, we studied the permutations that avoid two different patterns of length three. We focused on the distribution of peaks, valleys, double ascents, and double descents over these sets of permutations.

Definitions

- A **permutation** is an arrangement of the elements of a (finite) set. A set of n elements has $n!$ permutations, each of length n.
- A permutation p of length n contains a permutation q of length $m \le n$ as a **pattern** if there are m elements $p_{i_1}, p_{i_2}, \ldots, p_{i_m}$ in p with $i_1 < i_2 < \cdots < i_m$ such that $p_{i_a} < p_{i_b}$ if and only if $q_a < q_b$. Otherwise p avoids q.

Example 1. The permutation 1243 contains the pattern 132 because the elements 1, 4, and 3 relate to one another in the same way as 1, 3, and 2.

- The set of all permutations of length n is denoted S_n. The set of permutations of length n that avoids patterns x and y is called a **pattern class** and is denoted $S_n(x, y)$.

Example 2. The set $S_4(132, 321)$ contains all permutations that avoid the patterns 132 and 321.
$S_4(132, 321) = \{1234, 2134, 2314, 2341, 3124, 3412, 4123\}$.

In our research, we examined the following statistics on these permutations:

- An **ascent** is a set of 2 adjacent elements a_j and a_{j+1} in a permutation a such that $a_j < a_{j+1}$.
- A **descent** is a set of 2 adjacent elements a_j and a_{j+1} in a permutation a such that $a_j > a_{j+1}$.
- A **peak** is a set of 3 adjacent elements a_{j-1}, a_j, and a_{j+1} in a permutation a such that $a_j > a_{j+1}$ and $a_j > a_{j-1}$. A peak is equivalent to an ascent followed by a descent.
- A **valley** is a set of 3 adjacent elements a_{j-1}, a_j, and a_{j+1} in a permutation a such that $a_j < a_{j+1}$ and $a_j < a_{j-1}$. A valley is equivalent to a descent followed by an ascent.
- A **double ascent** is a set of 3 adjacent elements a_{j-1}, a_j, and a_{j+1} in a permutation a such that $a_{j-1} < a_j < a_{j+1}$.
- A **double descent** is a set of 3 adjacent elements a_{j-1}, a_j, and a_{j+1} in a permutation a such that $a_{j-1} > a_j > a_{j+1}$.

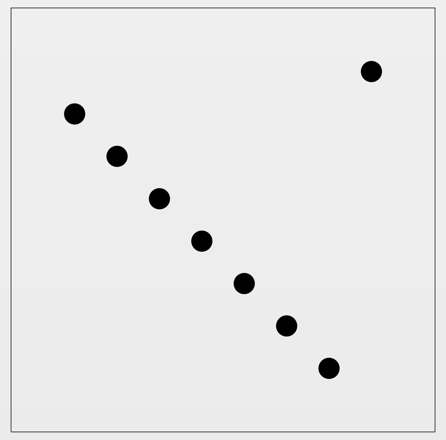
Results

For sufficiently large n, each formula will give the number of permutations of length n with the given statistic equal to k. Also, A076791 corresponds to an entry in The On-Line Encyclopedia of Integer Sequences (OEIS).

Patterns	Ascents	Descents	Double Ascents	Double Descents	Peaks	Valleys
123,132	$\binom{n}{2k}$	$\binom{n}{2(n-k-1)}$	trivial	$\binom{n-2}{k} + 2\binom{n-3}{k}$	$\binom{n}{2k+1}$	$2\binom{n-1}{2k}$
132,213	$\binom{n-1}{k}$	$\binom{n-1}{k}$	A076791	A076791	$\binom{n}{2k+1}$	$\binom{n}{2k+1}$
132,321	$\begin{cases} 1, & k = n-1 \\ \binom{n}{2}, & k = n-2 \end{cases}$	$\begin{cases} 1, & k = 0 \\ \binom{n}{2}, & k = 1 \end{cases}$	$\begin{cases} 1, & k = n-2 \\ n, & k = n-3 \\ \binom{n}{2} - n, & k = n-4 \end{cases}$	trivial	$\begin{cases} n, & k = 0 \\ \binom{n-1}{2}, & k = 1 \end{cases}$	$\begin{cases} 2, & k = 0 \\ \binom{n}{2} - 1, & k = 1 \end{cases}$
213,231	$\binom{n-1}{k}$	$\binom{n-1}{k}$	A076791	A076791	$\binom{n}{2k+1}$	$\binom{n}{2k+1}$
213,312	$\binom{n-1}{k}$	$\binom{n-1}{k}$	$\begin{cases} n, & k = 0 \\ \binom{n-1}{k+1}, & k \ge 1 \end{cases}$	$\begin{cases} n, & k = 0 \\ \binom{n-1}{k+1}, & k \ge 1 \end{cases}$	$\begin{cases} 2, & k = 0 \\ 2^{n-1} - 2, & k = 1 \end{cases}$	trivial

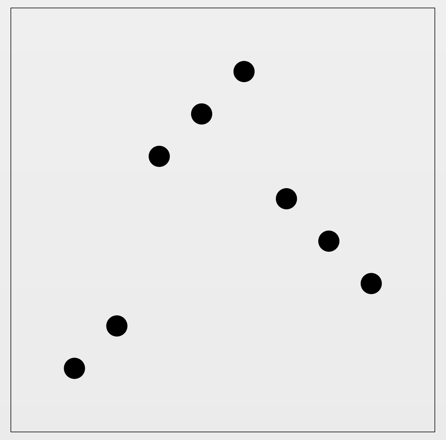
Diagrams and Examples

$S_8(123, 132)$



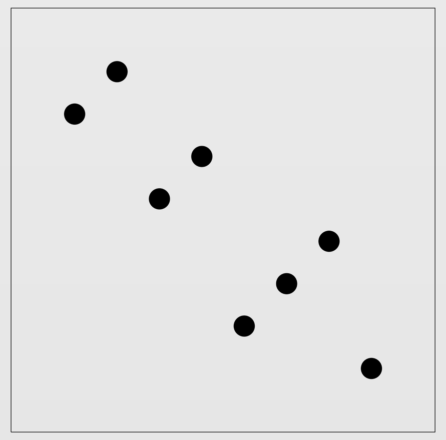
- The lowest digit in the permutation can either be placed in the last spot on the right, or in the first spot in from the right. This causes the permutation to have a diagonally downward facing pattern.

$S_8(213, 312)$



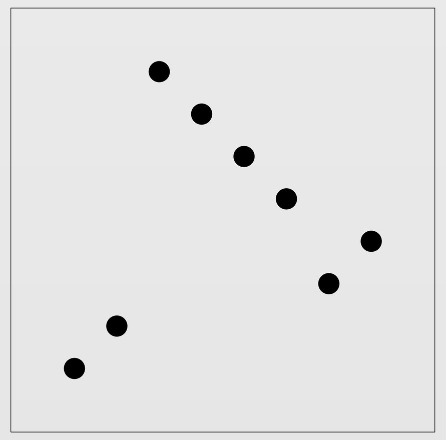
- There cannot be any valleys in the permutations produced. Thus, this causes only one peak to appear in the pattern since n has to be within the pattern. There are only two exceptions to this: when the patterns are completely ascending or completely descending.

$S_8(132, 213)$



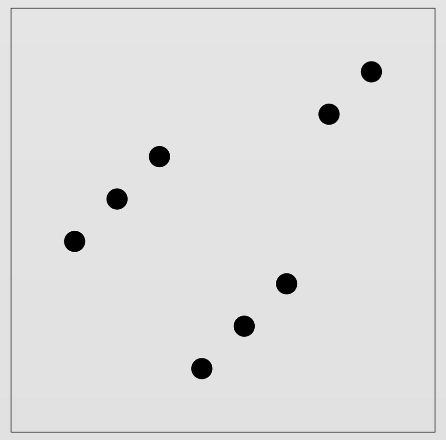
- To avoid both patterns, any consecutive 12 pattern must consist of consecutive values. If not, then we are guaranteed to have one of the two patterns. Hence, we see that each permutation looks like several consecutive increasing permutations next to each other.

$S_8(213, 231)$



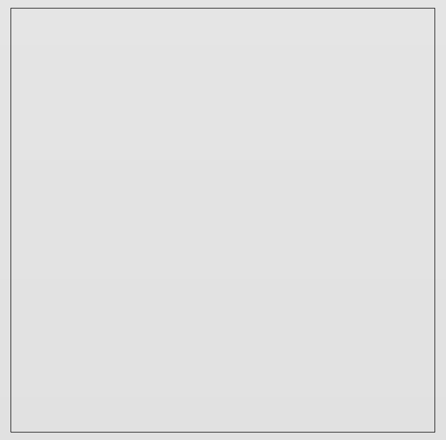
- Permutations that avoid (213, 231) must always start with the lowest number (in this case, 1) or the biggest number (in this case, 8). Everything below the last number increases toward it, and everything above the last number decreases toward it.

$S_8(132, 321)$



- Each permutation in $S_n(132, 321)$ must consist of one, two, or three ascending runs. If there are two ascending runs, they form the shape of a 21 pattern. If there are three ascending runs, they form the shape of a 213 pattern.

$S_8(123, 321)$



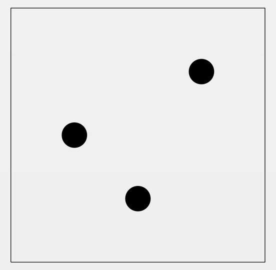
- This diagram is blank because permutations with $n \ge 5$ must contain at least one of these two patterns. Therefore, $S_8(123, 321)$ is empty.

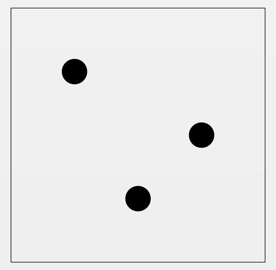
Wilf Equivalence

Two permutation classes are **trivially Wilf equivalent** if the patterns being avoided are rotations or reflections of one another, such as in $S_n(213, 231)$ and $S_n(312, 132)$. When we avoid two patterns of length 3, there are six permutation classes that are not trivially Wilf equivalent to each other, so we examined one representative from each class: $S_n(123, 132)$, $S_n(123, 321)$, $S_n(132, 213)$, $S_n(132, 321)$, $S_n(213, 231)$, and $S_n(213, 312)$. When we understand the distribution of statistics on one permutation class, then we understand the distributions on all trivially equivalent classes.

Example Proof

We will consider the pattern class $S_n(213, 312)$. These patterns are shown below.



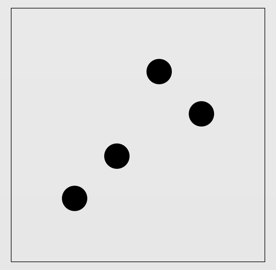


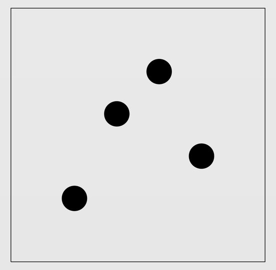
Because of the patterns we are avoiding, 1 will always be either the first or the last number in the permutation.

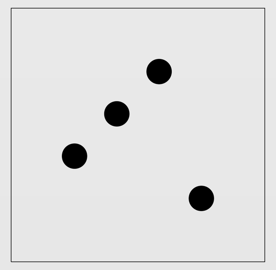
Theorem 1. $|\{\pi \in S_n(213, 312) | \pi \text{ has } k \text{ descents}\}| = \binom{n-1}{k}$.

Proof. We only need to choose the numbers to the right of the n. In order to get k descents, there must be k numbers to the right of n. □

Example. The number of permutations in $S_4(213, 312)$ with 1 descent is $\binom{n-1}{k} = \binom{3}{1} = 3$. The corresponding permutations are $\{1243, 1342, 2341\}$, and have graphs of:







References

For further reading, consult the following resources:

- R. Simion and F. Schmidt, Restricted Permutations, *European J. Combin.* **6** (1985), 383–406.
- N. J. A. Sloane, editor, The On-Line Encyclopedia of Integer Sequences, published electronically at `https://oeis.org`.