Chapter 7: Potential Energy and Voltage

Chapter Learning Objectives: After completing this chapter the student will be able to:

- Calculate the total energy contained in a system of discrete charges.
- Calculate the potential (voltage) between two points due to an electric field.
- Calculate the absolute potential in a region near discrete or continuous charges.

You can watch the video associated with this chapter at the following link:

Historical Perspective: Alessandro Volta (1745-1827) was an Italian physicist and chemist. He is best known as the inventor of the Voltaic pile, the earliest electrical battery that could provide a continuous electrical current to a circuit. In doing so, he debunked the theory that only living beings (such as electric eels) could generate electricity, and he founded the field of electrochemistry. The unit of voltage is named in his honor.

Photo credit: https://commons.wikimedia.org/wiki/File:Alessandro_Volta.jpeg, [Public domain], via Wikimedia Commons.
7.1 Mathematical Prelude: Gradient

Hopefully you will recall from section 5.2 that in the introduction to divergence, we also introduced the del operator. This definition is repeated below for your convenience:

\[ \nabla = \frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z \]  
(Copy of Equation 5.3)

Hopefully, you also recall that the divergence of a vector field \( \mathbf{A} \), which represents the presence of a source or sink of the vector \( \mathbf{A} \) at each point, is defined as “del dot \( \mathbf{A} \),” as shown below:

\[ \nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \]  
(Copy of Equation 5.4)

Similarly, we can calculate the gradient, or vector derivative of a function using the del operator. The gradient is a vector at each point that always points in the direction of the maximum rate of change of the function, and its magnitude is the rate of change in that direction. The gradient can be calculated as follows:

\[ \nabla a = \frac{\partial a}{\partial x} a_x + \frac{\partial a}{\partial y} a_y + \frac{\partial a}{\partial z} a_z \]  
(Equation 7.1)

Remember, the divergence converts a vector field into a set of scalar values that represent sources/sinks of the vector, while the gradient converts a scalar function into a vector field that represents the vector derivative of the function.

The gradient can also be expressed in cylindrical or spherical coordinates, as shown below:

\[ \nabla a = \frac{\partial a}{\partial \rho} a_\rho + \frac{1}{\rho} \frac{\partial a}{\partial \phi} a_\phi + \frac{\partial a}{\partial z} a_z \]  
(Equation 7.2)

\[ \nabla a = \frac{\partial a}{\partial r} a_r + \frac{1}{r} \frac{\partial a}{\partial \theta} a_\theta + \frac{1}{r \sin \theta} \frac{\partial a}{\partial \phi} a_\phi \]  
(Equation 7.3)

\[ \text{Example 7.1}: \text{ Given the following function for } f(x,y), \text{ calculate } \nabla f \text{ and evaluate it at } (0,0). \]

\[ f(x, y) = e^{-(x+y)} \]
Recall that in chapter 3 we first introduced Coulomb’s Law, which allows us to calculate the forces between two electric charges. From that concept of force, we also derived the mathematical abstraction of electric field, which allows us to consider the effect of a charge or region of charge on conceptual charges that could be placed at any point in the vicinity of that charge. (As an aside, we also saw that Coulomb’s Law can actually be derived from Gauss’s Law, which means that electric fields may be the more fundamental concept, and that the force between two charges are merely the manifestation of the electric field of one charge on the other.)

But what we have not yet considered is the potential energy that is caused when we bring electric charges near each other. This is a rich topic of consideration that will ultimately open up highly valuable relationships and concepts.

Imagine if the entire universe contained only one electric charge. In that case, the charge would never feel any electric forces, because there would be no other charges for it to interact with, as shown in Figure 7.1. This charge would also have no electric potential energy, since it is subject to no electric forces. (From here on, I will simply say it has no potential energy, although we will understand that I am neglecting gravitational potential energy for the sake of simplicity in this discussion.)

![Figure 7.1. A single isolated charge has no electric potential energy.](image)

If a second electric charge is brought into the vicinity of the first charge, then there will be electrical force between the two charges. This electric force would be zero when the two charges are infinitely far apart, but it will increase as they get closer.

![Figure 7.2. Bringing a second charge near the first one creates potential energy.](image)

How much potential energy is present between these two electric charges? Recall the relationship between force and energy, as shown below:
\[ \Delta W = - \int_a^b \mathbf{F} \cdot d\mathbf{l} = -Q \int_a^b \mathbf{E} \cdot d\mathbf{l} \]  
(Equation 7.4)

The negative signs in this equation come from the fact that if \( \mathbf{F} \) and \( d\mathbf{l} \) are both pointing in the same direction, \( \mathbf{F} \cdot d\mathbf{l} \) will be positive, but that would indicate motion in the direction of a force (like rolling a ball down a hill), which will reduce potential energy. Increasing potential energy will require motion in the opposite direction of the force (like pushing a ball uphill), and this will yield a negative dot product.

We can now plug Coulomb’s Law into equation 7.4 and calculate the precise value of the energy present in this system of two charges located at \( x_a \) and \( x_b \):

\[ \Delta W = - \int_{-\infty}^{x_b} \frac{Q_1 Q_2}{4\pi \varepsilon_0 (x - x_a)^2} dx = \frac{Q_1 Q_2}{4\pi \varepsilon_0 |x_b - x_a|} \]  
(Equation 7.5)

Notice that this same result will be obtained whether we think of charge \( Q_2 \) approaching an immobile charge \( Q_1 \) or vice versa, but we will think of it as charge \( Q_2 \) moving toward charge \( Q_1 \). Just as we did with electric charge, we can now consider the effect that \( Q_1 \) is having on its environment and the potential energy that would be required to move a test charge \( Q_2 \) to any point in its vicinity. We will do this by defining a new quantity \( V \) (called “potential” or “voltage” that includes everything except \( Q_2 \) from equation 7.5:

\[ \boxed{V_1 = - \int_{-\infty}^{x_b} E_1(x) dx = - \int_{-\infty}^{x_b} \frac{Q_1}{4\pi \varepsilon_0 (x - x_a)^2} dx = \frac{Q_1}{4\pi \varepsilon_0 |x_b - x_a|}} \]  
(Equation 7.6)

Given this definition, we could then easily calculate the potential energy as the product of the voltage due to the first charge multiplied by the charge of the second charge:

\[ \Delta W = V_1 \cdot Q_2 \]  
(Equation 7.7)

Notice that “potential” and “potential energy” are related but separate quantities. Potential energy is the stored energy in the system that can be converted to other forms (such as kinetic energy), whereas potential is the quantity that exists at every point around a charge or collection of charges that will create potential energy if another charge is brought into that region.

As described above, it doesn’t matter which charge we think of as being brought into the vicinity of the other, so it is just as valid to state that:
\[ \Delta W = V_2 \cdot Q_1 \]  

(Equation 7.8)

Given that \( V_2 \) is calculated as:

\[ V_2 = -\int_{-\infty}^{x_a} E_2(x) \, dx = \frac{Q_2}{4\pi \varepsilon_0 |x_b - x_a|} \]  

(Equation 7.9)

Thus, we can see that there are two components to the energy present between two charged particles. The first is the voltage caused by one charge, and the second is the amount of charge that is being moved into the region. This is very similar to thinking about the energy it takes to carry a rock up a hill. One factor is the height of the hill (like the voltage), and the second factor is the size of the rock (like the amount of charge being moved in).

Now, let’s bring a third charge into the vicinity of the first two:

![Figure 7.3. Bringing a third charge near the first two adds more potential energy.](image)

The third charge is affected by both of the first two charges, as shown below:

\[ \Delta W_3 = -\int_{-\infty}^{x_c} \frac{Q_1 Q_3}{4\pi \varepsilon_0 (x - x_a)^2} \, dx - \int_{-\infty}^{x_c} \frac{Q_2 Q_3}{4\pi \varepsilon_0 (x - x_b)^2} \, dx \]  

(Equation 7.10)

These integrals can be solved as follows:

\[ \Delta W_3 = \frac{Q_1 Q_3}{4\pi \varepsilon_0 |x_c - x_a|} + \frac{Q_2 Q_3}{4\pi \varepsilon_0 |x_c - x_b|} \]  

(Equation 7.11)

Plugging in our definition of the voltage created by a point charge (Equation 7.9), we get:

\[ \Delta W_3 = Q_3 \cdot V_1 + Q_3 \cdot V_2 \]  

(Equation 7.12)

Now considering the total electrostatic energy contained in the system of three charges, we get:

\[ \Delta W_{total} = W_1 + W_2 + W_3 = 0 + Q_2 V_{12} + Q_3 (V_{13} + V_{23}) \]  

(Equation 7.13)

If we were to generalize this to \( N \) charges, the equation could be written as:
And if we define:

\[
V_i = \sum_{j=1}^{N(j \neq i)} \frac{Q_j}{4\pi \epsilon_0 |x_{ij}|}
\]  
(Equation 7.15)

Then:

\[
W_e = W_{total} = \frac{1}{2} \sum_{i=1}^{N} Q_i V_i
\]  
(Equation 7.16)

The ½ in equations 7.14 and 7.16 is to prevent the energy from being double-counted. The way the summation is written, the energy between, say, charges 5 and 7 will be counted once as charge 5 affecting charge 7, but it is also counted a second time as charge 7 affecting charge 5. Since this is the same electrostatic energy counted twice, we simply divide the summation by two to eliminate the double-counting.

If we generalize Equation 7.16 to allow for continuous charge, the summation becomes an integral over volume, and we get:

\[
W_e = \frac{1}{2} \int_{\Delta v} \rho_v V dv
\]  
(Equation 7.17)

Combine Gauss’s Law with Equation 7.17, and we find:

\[
W_e = \frac{1}{2} \int_{\Delta v} \rho_v V dv = \frac{1}{2} \int_{\Delta v} (\epsilon_0 \nabla \cdot \mathbf{E}) V dv
\]  
(Equation 7.18)

Simplifying this equation, we see that:

\[
W_e = \frac{\epsilon_0}{2} \int_{\Delta v} E^2 dv
\]  
(Equation 7.19)

Example 7.2: How much electric potential energy is contained in a spherical shell of inner radius 0.5m and outer radius 1m centered on a point charge of 2C?
7.3 Voltage

In the derivation in the previous section, we saw that voltage (also called potential) is a measure of how much energy it takes to move a positive test charge into the vicinity of an existing charge or set of charges.

There are two ways we can talk about voltage: absolute and relative. Absolute voltage is the amount of energy it would take to move a test charge into the region from infinitely far away. We have already seen this calculation, but we didn’t have a name for it then:

\[
V_2 = - \int_{-\infty}^{x_a} E_2(x) \, dx = \frac{Q_2}{4\pi\epsilon_0 |x_b - x_a|}
\]

(Copy of Equation 7.9)

Notice that we are implicitly taking “the test charge is infinitely far away” as our reference point for this quantity. All voltages must have a reference. Just as we can’t specify an altitude or elevation without specifying a reference (such as “Mount Everest is 29,029 feet above sea level”), we can’t specify a voltage without specifying a reference. For absolute voltage, the reference is when the test charge is infinitely far away.

In general, we will also refer to the reference voltage as the “ground voltage” or “ground potential.” For conceptual problems like the ones in this class, the ground voltage is a conceptual case where the test charge is infinitely far away. In the real world, we don’t have the luxury of moving charges infinitely far away, so we take the practical approach of choosing the Earth’s surface as the reference point. Soil is generally a very good conductor, so the Earth’s surface is typically all at approximately the same voltage, and it is conveniently located everywhere. We drive metal stakes ten feet into the Earth’s surface, and that is then used as the ground voltage.

The other way we can specify voltage is as a relative quantity between two points. For example, we could say that point A is 3V higher than point B. This is similar to saying that Denver, Colorado is 4486 feet higher than Valparaiso, Indiana. The relative voltage can be calculated by subtracting the absolute voltages of the two points, just as the elevation difference between Denver and Valpo can be calculated by subtracting their elevations above sea level (5280 – 794 = 4486).
\[ \Delta V_{ab} = V_a - V_b = \frac{1}{Q} \left\{ \int_{-\infty}^{a} \mathbf{F} \cdot d\mathbf{l} - \int_{-\infty}^{b} \mathbf{F} \cdot d\mathbf{l} \right\} \]  \hspace{1cm} \text{(Equation 7.20)}

\[ \Delta V_{ab} = \frac{1}{Q} \left\{ Q \int_{-\infty}^{a} \mathbf{E} \cdot d\mathbf{l} - Q \int_{-\infty}^{b} \mathbf{E} \cdot d\mathbf{l} \right\} \]  \hspace{1cm} \text{(Equation 7.21)}

\[ \Delta V_{ab} = \int_{a}^{b} \mathbf{E} \cdot d\mathbf{l} \]  \hspace{1cm} \text{(Equation 7.22)}

\( \textbf{Example 7.3:} \) What is the relative voltage \( V_{AB} \) between point A, which is 1 meter away from a 5C charge, and point B, which is 3 meters away from the same charge? What is the relative voltage \( V_{AC} \) between two points that are both one meter away from the charge but in different directions?

Points A and C in the previous example are on what is known as an “equipotential surface” because all points on a circle the same radius from a point charge all have the same potential. As we will see, all conductors form equipotential surfaces, which means that this idea is very important to us.

Interestingly, the path followed when moving from point A to point B does not affect the result. As shown in Figure 7.4, only the portion of the path that is parallel to the electric field will
contribute toward the voltage difference. If the path swings out so that part of it is perpendicular to the electric field, this does not affect the voltage. Just like your elevation increase is the same whether you walk straight up a hill or follow a path with many switchbacks or spiral around the hill, only the portion of your path going uphill (parallel to the direction of gravity) will change your elevation.

Figure 7.4. Traveling different paths between two points yields the same relative voltage.

\( V_{AB} \) could be described as “how much higher point A’s voltage is than point B’s voltage.” By reversing the subscripts on Equation 7.22, we find that:

\[
V_{BA} = \int_{b}^{a} E \cdot dl = -V_{AB}
\]

(Equation 7.23)

Rearranging this equation, we find:

\[
V_{AB} + V_{BA} = 0
\]

(Equation 7.24)

This equation says that if we follow a path from A to B and then follow a path from B back to A, the total voltage drop will be zero. We can add additional “stops” along the closed loop, which gives us:

\[
\sum_{i=1}^{N} V_i = 0
\]

(Equation 7.25)

Or we can write it in continuous form, which gives:

\[
\oint E \cdot dl = 0
\]

(Equation 7.26)

If Equation 7.25 looks familiar, it should. That is the discrete form of Kirchhoff’s Voltage Law, which you have seen before, and Equation 7.26 is the continuous form, which we will use in this class.
Example 7.4: What is the voltage at all points in the region between two concentric hollow cylinders of radius \(a\) and \(b\) if the potential of the inner cylinder is \(V_0\) and the potential of the outer cylinder is zero?

Since the relative voltage between two points is the integral of the electric field between them, the electric field between two nearby points can be approximated as follows:

\[
\begin{align*}
   dV & \approx -E \cdot dl
\end{align*}
\]

(Equation 7.27)
The negative sign is because the voltage decreases if motion is in the direction of the electric field (going “downhill” or “downstream,” and it increases if motion is in the opposite direction of the electric field (going “uphill” or “upstream.”)

Plugging in the equation for $dl$ in rectangular coordinates (Equation 2.6), we find:

$$dV = -E_x a_x dx - E_y a_y dy - E_z a_z dz$$ \hspace{2cm} (Equation 7.28)

The approximate equality is made an equality if we assume that the two points under consideration are infinitesimally close to each other.

By the chain rule, we know that:

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz$$ \hspace{2cm} (Equation 7.29)

Then, comparing terms from Equations 7.28 and 7.29, we can derive the following equation:

$$\mathbf{E} = -\frac{\partial V}{\partial x} \mathbf{a}_x - \frac{\partial V}{\partial y} \mathbf{a}_y - \frac{\partial V}{\partial z} \mathbf{a}_z$$ \hspace{2cm} (Equation 7.30)

Recalling our just-in-time discussion of the gradient (vector derivative), we see that:

$$\mathbf{E} = -\nabla V$$ \hspace{2cm} (Equation 7.31)

**Example 7.5:** If $V(x,y,z)=x^2+y+e^{-z}$, what is the electric field that is causing this voltage?

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7.5 **Poisson’s Equation and Laplace’s Equation**

As we saw in both lesson 5, the point form of Gauss’s Law can be written as:

$$\nabla \cdot \mathbf{E} = \frac{\rho_v}{\epsilon_0}$$ \hspace{2cm} (Copy of Equation 5.6)

Now if we substitute Equation 7.31 into this equation, we find:
\[ \nabla \cdot (-\nabla V) = \frac{\rho_v}{\varepsilon_0} \]  
\text{(Equation 7.32)}

Moving the negative sign to the right side, we see that:

\[ (\nabla \cdot \nabla) V = -\frac{\rho_v}{\varepsilon_0} \]  
\text{(Equation 7.33)}

And introducing a new operator, the Laplacian (represented as \( \nabla^2 \)), to represent \( \nabla \cdot \nabla \), we obtain what is known as Poisson’s Equation:

\[ \nabla^2 V = -\frac{\rho_v}{\varepsilon_0} \]  
\text{(Equation 7.34)}

If we assume that the region of space under consideration has no charge density, then the right side of Poisson’s Equation goes to zero, and we find:

\[ \nabla^2 V = 0 \]  
\text{(Equation 7.35)}

This is Laplace’s Equation, and it is only valid where \( \rho_v = 0 \).

The Laplacian operator (\( \nabla^2 \)) can be represented as follows:

\[ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \]  
\text{(Equation 7.36)}

The Laplacian of \( V \) can be calculated as:

\[ \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \]  
\text{(Equation 7.37)}

\textbf{Example 7.6:} If \( V(x,y,z) = xy^2 + z^3 \), what is the charge density that is causing this voltage?
Like most important results in this class, Poisson’s Equation can be written in either point form (shown in Equation 7.34), or in integral form, as shown below.

\[ V(x, y, z) = \frac{1}{4\pi \varepsilon_0} \int_{\Delta v} \frac{\rho_v(x', y', z')}{R} dx' dy' dz' \]  

(Equation 7.38)

where:

\[ R = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2} \]  

(Equation 7.39)

This equation is not really feasible to use for calculating results analytically by hand, but it is very useful when performing numerical calculations using the computer, as we will see.

### 7.6 Summary

In this chapter, we have derived many very important equations. It can sometimes be confusing to remember which equation to use in each circumstance. The following table can be useful for choosing the correct equation. A few of these equations were taken from previous lessons, but most of them were introduced and derived today.

<table>
<thead>
<tr>
<th>Have:</th>
<th>( \rho_v )</th>
<th>( \mathbf{E} )</th>
<th>( V )</th>
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<tbody>
<tr>
<td>Need:</td>
<td>( \rho_v )</td>
<td>( \nabla \cdot \mathbf{E} = \frac{\rho_v}{\varepsilon_0} )</td>
<td>( \nabla^2 V = -\frac{\rho_v}{\varepsilon_0} )</td>
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<td>( \mathbf{E} )</td>
<td>[ E = \frac{1}{4\pi \varepsilon_0} \int_{\Delta v} \frac{\rho_v(x', y', z')}{R^2} dV' ]</td>
<td>( \mathbf{E} = -\nabla V )</td>
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<td>( V )</td>
<td>[ V(x, y, z) = \frac{1}{4\pi \varepsilon_0} \int_{\Delta v} \frac{\rho_v(x', y', z')}{R} dx' dy' dz' ]</td>
<td>( \Delta V_{ab} = \int_a^b \mathbf{E} \cdot d\mathbf{l} )</td>
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<tr>
<td>( W_e )</td>
<td>[ W_e = \frac{1}{2} \int_{\Delta v} \rho_v V dV ]</td>
<td>[ W_e = \frac{\varepsilon_0}{2} \int_{\Delta v} E^2 dV ]</td>
<td>[ W_e = \frac{1}{2} \int_{\Delta v} \rho_v V dV ]</td>
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