Chapter 5: Gauss’s Law I

Chapter Learning Objectives: After completing this chapter the student will be able to:

- Calculate a surface integral.
- Calculate the divergence of a vector field.
- Use Gauss’s Law to calculate the electric field in the vicinity of a highly symmetric hollow three-dimensional object.

You can watch the video associated with this chapter at the following link:

Historical Perspective: Carl Friedrich Gauss (1777-1855) was one of the greatest mathematicians and physicists in history. He made significant contributions to algebra, astronomy, electrostatics, geophysics, magnetic fields, matrix theory, number theory, optics, and statistics. In addition to Gauss’s Law, his name is attached to a unit of magnetic field and the most important statistical distribution (the normal, or “Gaussian” distribution).

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Mathematical Prologue #1: Surface Integrals

There are a handful of mathematical skills you will need to use in the class, and I will attempt to introduce each one just in time for you to use it. Today, will need to remember (or learn) how to perform a surface integral. At its simplest, a surface integral can be thought of as the quantity of a vector field that penetrates through a given surface, as shown in Figure 5.1.

Figure 5.1. Schematic representation of a surface integral

The surface integral is calculated by taking the integral of the dot product of the vector field with the vector $\mathbf{dS}$ that is perpendicular (at every point) to the surface being considered. This calculation is shown in Equation 5.1.

$$\iiint_{\Delta S} \mathbf{F} \cdot \mathbf{dS}$$  \hspace{1cm} \text{(Equation 5.1)}

The use of a dot product allows us to calculate the portion of the vector field $\mathbf{F}$ that is pointing in the same direction as $\mathbf{dS}$. In Figure 5.1, these vectors are parallel, so it is 100% aligned. But this is not always the case, as shown in Figure 5.2. Here, the vectors are at an angle to the surface, so only a portion of the vector penetrates through the surface.

Figure 5.2. The vector field is not necessarily perpendicular to the surface.
Although only one vector $\mathbf{dS}$ is shown in Figures 5.1 and 5.2, this is a simplification. If the surface is curved, this $\mathbf{dS}$ vector will be different at each point along the surface, as shown in Figure 5.3.

![Figure 5.3. The $\mathbf{dS}$ vector can vary from point to point on a curved surface (side view).](image)

Fortunately, these complexities will all be handled for us automatically when we use the dot product and the integral.

Very often, the most important type of surface integral is over a closed surface. This is so significant that we have a special symbol to represent a surface integral over a closed surface, as shown in Equation 5.2.

\[
\oint_{\Delta S} \mathbf{F} \cdot \mathbf{dS}
\]  

(Equation 5.2)

When working with a closed integral, the vector $\mathbf{dS}$ always points outward from the closed surface. If the closed surface has flat sides, as shown in Figure 5.4, you will need to perform an integral over each of the sides.

![Figure 5.4. The six sides of a cube each have their own surface vector.](image)
Example 5.1: Calculate the surface integral of the following vector field with the surface shown:

\[ \mathbf{F}(x, y, z) = 2a_x + 3a_y + 4a_z \]

Example 5.2: Calculate the surface integral of the following vector field with the surface shown:

\[ \mathbf{F}(x, y, z) = ya_x + za_y + xa_z \]

Example 5.3: Calculate the closed surface integral of the following vector field with the cube shown:

\[ \mathbf{F}(x, y, z) = xa_x + ya_y + 2a_z \]
Surface integrals can also be calculated using cylindrical and spherical coordinates. Recall the differential surface terms we first saw in lesson 2. (I promised they would come in handy eventually!)

\[
\begin{align*}
\mathbf{ds}_\rho &= \pm \rho \cdot d\phi \cdot dz \cdot \mathbf{a}_\rho \\
\mathbf{ds}_\phi &= \pm d\rho \cdot dz \cdot \mathbf{a}_\phi \\
\mathbf{ds}_z &= \pm \rho \cdot d\rho \cdot d\phi \cdot \mathbf{a}_z
\end{align*}
\]

\[\text{(Copy of Equation 2.9)}\]
\[\text{(Copy of Equation 2.10)}\]
\[\text{(Copy of Equation 2.11)}\]

\[
\begin{align*}
\mathbf{ds}_r &= \pm r^2 \sin \theta \cdot d\theta \cdot d\phi \cdot \mathbf{a}_r \\
\mathbf{ds}_\theta &= \pm r \sin \theta \cdot dr \cdot d\phi \cdot \mathbf{a}_\theta \\
\mathbf{ds}_\phi &= \pm r \cdot dr \cdot d\theta \cdot \mathbf{a}_\phi
\end{align*}
\]

\[\text{(Copy of Equation 2.25)}\]
\[\text{(Copy of Equation 2.26)}\]
\[\text{(Copy of Equation 2.27)}\]

**Example 5.4:** Calculate the closed surface integral of the following vector field in spherical coordinates with a sphere of radius 1 centered at the origin.

\[\mathbf{A}(r) = \frac{1}{4\pi r^2} \mathbf{a}_r\]
The divergence of a vector field is written most compactly as the dot product between the “del” operator and the vector field being considered. The del operator is used in a variety of vector calculus expressions, and it is defined as follows:

\[ \nabla = \frac{\partial}{\partial x} \mathbf{a}_x + \frac{\partial}{\partial y} \mathbf{a}_y + \frac{\partial}{\partial z} \mathbf{a}_z \]  

(Equation 5.3)

Notice that the del operator by itself doesn’t make very much sense, but if we “dot” it with a vector, we get the following result:

\[ \nabla \cdot \mathbf{A} = \left( \frac{\partial}{\partial x} \mathbf{a}_x + \frac{\partial}{\partial y} \mathbf{a}_y + \frac{\partial}{\partial z} \mathbf{a}_z \right) \cdot \left( A_x \mathbf{a}_x + A_y \mathbf{a}_y + A_z \mathbf{a}_z \right) \]

\[ \nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \]  

(Equation 5.4)

Although this could be simply known as “del dot A,” we will refer to this as “the divergence of A.” The divergence of A is equal to the surface integral of a closed surface that is infinitesimally small (in the limit as the volume of the closed surface approaches zero) as shown in Equation 5.5:

\[
\text{div} \mathbf{A} = \nabla \cdot \mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint \mathbf{A} \cdot d\mathbf{S}}{\Delta v}
\]  

(Equation 5.5)

Thinking about what this means in real terms, the divergence of A represents whether a certain point contains a “source” to create the vector quantity or a “sink” to absorb it. If there is a source of the vector field (such as a positive electric charge is a source of electric fields), then the divergence will be positive. If there is a sink of the vector field (such as a negative electric charge is a sink of electric fields), then the divergence will be negative. If the divergence is zero at a point, then there is neither a source nor a sink of the vector field at that point.

**Example 5.5:** Determine the divergence of the following function at the origin (0,0,0). Is there a source or sink of the vector field at that point?

\[
\mathbf{A}(x, y, z) = 3xy \mathbf{a}_x + y \mathbf{a}_y + -4xy e^{-z} \mathbf{a}_z
\]
5.3 Introduction to Gauss's Law

Now that we have covered the necessary mathematical tools, Gauss’s Law itself is almost trivial. Remembering that electric charges are sources/sinks of electric fields, and knowing that the divergence of a vector field can be used to calculate the sources/sinks of a generic field, we can set the divergence equal to the charge density, as shown in equation 5.6.

\[ \nabla \cdot \mathbf{E} = \frac{\rho V}{\varepsilon_0} \]  

(Equation 5.6)

This is known as the “Point Form of Gauss’s Law” because the divergence allows us to perform the equation on a point-by-point basis. The \( \varepsilon_0 \) is included in the equation because of the units we have chosen, and it indicates the amount of electric fields that are generated/absorbed by a unit of charge.

There is another form of Gauss’s Law, known as the “Integral Form of Gauss’s Law.” This form of the law, shown in Figure 5.7, considers not just a point in space, but an entire region of space.

\[ \oint \mathbf{E} \cdot d\mathbf{S} = \frac{Q_{\text{enc}}}{\varepsilon_0} \]  

(Equation 5.7)

The integral form of Gauss’s says that the source of any electric field lines leaving a region must be caused by charges that are enclosed (\( Q_{\text{enc}} \)) within that region. Think about it this way: If you see a cave entrance with a light coming out of it, then you logically know that there must be a light source within that cave. That’s what Gauss’s Law allows us to say for electric fields.

We will tend to use the integral form of Gauss’s Law in this class, but the point form can also be useful. **The most important thing to know about Gauss’s Law is that it is primarily (only) useful in cases where there is a very high level of symmetry in the geometry of the problem.** Fortunately, this will be true for most of our problems in this class.

We will want to enclose (or partially enclose) the regions of charge density inside a closed surface so that we can perform a closed-surface integral as required in Equation 5.7. Such a closed surface is so important that we often refer to it as a “Gaussian” surface. A Gaussian surface has no other special characteristics other than being a closed surface.
You should always choose a Gaussian surface such that the electric field across each side is either constant or zero. To do that, you will want to choose a surface whose symmetry matches the symmetry of the problem itself. In those cases, the integral form of Gauss’s Law reduces to:

\[ E \cdot A = \frac{Q_{enc}}{\varepsilon_0} \]  
(Equation 5.8)

Notice that a calculation that would have originally required either the use of partial derivatives (for the point form) or surface integrals (for the integral form) can now be performed using simple multiplication and division. This is the power of symmetry and choosing the right Gaussian surface!

Thus, what we will usually find when applying Gauss’s Law is that we are given a charge density (a sphere, cylinder, plane, or some other highly symmetric shape), and we are asked to find the electric field. This problem then typically reduces to selecting the right Gaussian surface, calculating the total charge enclosed within the surface, and applying the following modification of Equation 5.8:

\[ E = \frac{Q_{enc}}{A \cdot \varepsilon_0} \]  
(Equation 5.9)

5.4 Gauss’s Law Examples with Hollow Shapes

There are two main types of problems that can be solved using Gauss’s Law: those with hollow shapes and those with solid shapes. Of course, in the real world, there are no truly hollow shapes, but some shapes have very thin walls, and it is reasonable to approximate them as being hollow. Still, solid shapes are the best representation of the real world. We’ll finish today considering the (simpler) hollow shapes and will tackle a variety of problems with solid shapes in chapter 6.

Example 5.5: Consider a hollow sphere of radius a and a total charge Q spread across the entire surface of the sphere. What is the electric field at any point inside the sphere (r < a)?
Example 5.6: Given the sphere from the previous example, what is the electric field at any point outside the sphere (r > a)?

Example 5.7: What is the electric field at a distance z above an infinite plane of charge with a charge density of $\rho_s$?

Example 5.8: What is the electric field in between two infinite planes of charge, one with a charge density of $\rho_s = +2\text{C/m}^2$ and the other with a charge density $\rho_s = -2\text{C/m}^2$.
Example 5.9: What is the electric field at any point outside an infinitely long hollow cylinder of radius $a$ with a linear charge density of $\rho_L$?

5.5 Summary

- A surface integral is used to calculate the quantity of a vector field that is penetrating through a given surface. It is calculated by integrating the dot product of the vector field with a vector that is perpendicular to the surface at every point. There is a special symbol for a surface integral over a closed surface.

$$ \iint_{\Delta S} \mathbf{F} \cdot d\mathbf{S} \quad \oint_{\Delta S} \mathbf{F} \cdot d\mathbf{S} $$

- The divergence of a vector field can be used to determine whether a source or sink of a vector field exists at a particular point. The divergence can be represented using the del operator, and it is calculated by summing the partial derivative of each of the three components of the vector field.

$$ \nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} $$

- Gauss’s Law (in point form or integral form) allows us to determine an electric field when given a region of charge.

$$ \nabla \cdot \mathbf{E} = \frac{\rho_V}{\varepsilon_0} \quad \oint \mathbf{E} \cdot d\mathbf{S} = \frac{Q_{enc}}{\varepsilon_0} $$

- If we choose the Gaussian surface such that each side has an electric field that is either constant or zero, the integral reduces to a multiplication.

$$ E = \frac{Q_{enc}}{A \cdot \varepsilon_0} $$