Chapter Learning Objectives: After completing this chapter the student will be able to:

- Calculate the input impedance, half-power beam width, directivity, gain, and effective area of an antenna.
- Use the Friis equation to find power available at the output terminals of a receiving antenna.

You can watch the video associated with this chapter at the following link: [VIDEO]

Historical Perspective: Edwin Hubble (1889-1953) was an American astronomer who used radio telescopes to discover galaxies outside the Milky Way and to prove that the universe is expanding by observing the red shift of distant galaxies. The Hubble Space Telescope is named after him.

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31.1 Input Impedance

We spent quite a lot of time earlier discussing how to match the impedance of a transmission line to its load. During those discussions, we tended to think of the load as unchangeable, provided to us by a customer, and we had to find a way to match it to the line. But now our focus is shifting, and we see that the load is often an antenna. This, of course, raises the question, “What is the input impedance of an antenna?

Figure 31.1 shows an AC source $V_s$ connected to an antenna with input impedance $Z_L$ through a transmission line of length $L$ with characteristic impedance $Z_C$.

If we draw a sphere around the antenna and integrate the power radiating through the sphere, this will give the total power radiated away from the antenna. To the transmission line, this radiated power looks exactly like heat dissipated in a resistor, so we can use that total power to calculate the input impedance (also known as the “radiation resistance”). We can relate total radiated power to input impedance as follows:

\[ P_{\text{rad}} = I_{\text{RMS}}^2 \cdot R_{\text{in}} = \frac{1}{2} I_0^2 R_{\text{in}} \quad \text{(Equation 31.1)} \]

This can be rearranged to:

\[ R_{\text{in}} = 2 \frac{P_{\text{rad}}}{I_0^2} \quad \text{(Equation 31.2)} \]

Let’s calculate the input impedance of an infinitesimal dipole antenna. In Chapter 29, we calculated the total power radiated by an infinitesimal dipole:

\[ P_{\text{rad}} = Z_0 \frac{(k I_0 L)^2}{12\pi} \quad \text{(Copy of Equation 29.47)} \]
Incorporating the fact that $Z_0=120\pi$ and $k=2\pi/\lambda$, this becomes:

$$P_{rad} = \frac{120\pi \left(\frac{2\pi}{\lambda} I_0 \mathcal{L}\right)^2}{12\pi}$$  \hspace{1cm} (Equation 31.3)

Simplifying this equation gives:

$$P_{rad} = 40\pi^2 \left(\frac{\mathcal{L}}{\lambda} I_0\right)^2$$  \hspace{1cm} (Equation 31.4)

Substituting Equation 31.4 into Equation 31.2, we find:

$$R_{in} = \frac{2 \cdot 40\pi^2 \left(\frac{\mathcal{L}}{\lambda} I_0\right)^2}{I_0^2}$$  \hspace{1cm} (Equation 31.5)

This, in turn simplifies as follows:

$$R_{in} = 80\pi^2 \left(\frac{\mathcal{L}}{\lambda}\right)^2$$  \hspace{1cm} (Equation 31.6)

**Example 31.1:** Determine the input impedance of an antenna that is 1\% of the wavelength. How much current would have to be provided to this antenna in order to radiate a total of one watt of power?

This equation is plotted below in Figure 31.2. Notice how low the input impedance of the infinitesimal antenna is. This is one of the reasons why it is very difficult to get good power transmitted by this antenna—when the load impedance is this low, it is very difficult to match it to a transmission line.
Similarly, we can prove that the input impedance of a small loop antenna is:

\[ R_{in} = 20\pi^2 (k\alpha)^4 \]  

(Equation 31.7)

Calculating the input impedance of a finite dipole is much more challenging. It requires integrating the average Poynting vector, shown in Equation 30.13, in spherical coordinates:

\[ \mathbf{S}_{av}(r) = \frac{1800I_m^2}{Z_0r^2} \cdot \frac{\cos\left(\frac{k\ell}{2}\cos\theta\right) - \cos\left(\frac{k\ell}{2}\right)}{\sin(\theta)}^2 \mathbf{a}_r \]  

(Copy of Equation 30.13)

We will omit the steps to solve this integral, but we will include the result:

\[ R_{in} = \frac{Z_0}{2\pi} \left[ C + ln(k\mathcal{L}) - C_i(k\mathcal{L}) + \frac{1}{2} \sin(k\mathcal{L}) \cdot \left( S_i(2k\mathcal{L}) - 2S_i(k\mathcal{L}) \right) + \frac{1}{2} \cos(k\mathcal{L}) \cdot \left( C + ln(k\mathcal{L}/2) + C_i(2k\mathcal{L}) - 2C_i(k\mathcal{L}) \right) \right] \]  

(Equation 31.8)

Where:

\[ C = 0.5772 \]  

(Equation 31.9)

\[ C_i(x) = \int_x^\infty \frac{\cos(y)}{y} dy \]  

(Equation 31.10)
Equation 31.10, which is known as a “cosine integral,” can be evaluated numerically in Matlab using the function \texttt{cosint(x)}, and Equation 31.11, which is known as a “sine integral,” can be evaluated in Matlab using the function \texttt{sinint(x)}. Equation 31.8 can be evaluated for a particular antenna length, and Figure 31.3 shows the result of this calculation. In practice, it is more reasonable to look up a value of input impedance in this figure rather than calculating it from Equation 31.8 unless you require an extraordinarily high degree of precision.

\[
S_i(x) = \int_0^x \frac{\sin(y)}{y} \, dy \tag{Equation 31.11}
\]

Notice that all of Figure 31.2 lies in the first horizontal division of Figure 31.3.

\textbf{Example 31.2}: What is the input impedance of a half-wavelength antenna?

\textbf{Example 31.3}: You are using a transmission line with a characteristic impedance of 100Ω to transmit a signal with a wavelength of 3m. What is the shortest antenna that would be impedance matched with this line?
31.2 Directivity and Half-Power Beam Width

Directivity (also known as “directive gain”) is a parameter that measures the degree to which the radiation emitted by an antenna is concentrated in a single direction. There are applications where a very high directivity is quite desirable, but there are others where we would prefer the antenna to be transmitting equally in all directions (also known as an isotropic antenna). For example, if you are using an antenna to send out and receive a radar pulse to detect the position of a storm or an airplane, then the more directivity your antenna has, the more precise the location of the object will be. But if you are designing a broadcast television antenna, you would probably prefer it to transmit equally in all directions.

The directivity of an antenna is defined mathematically as the ratio of the maximum radiation intensity that an antenna creates in a particular direction divided by the average radiation intensity over all directions (as if the antenna were isotropic). The average radiation can then be calculated as the total radiated power divided by the surface area of an enclosing surface such as a sphere.

\[
D = \frac{I(\theta, \phi)_{MAX}}{I(\theta, \phi)_{AVERAGE}} = \frac{I(\theta, \phi)_{MAX}}{P_{RAD}/A}
\]

(Equation 31.12)

The radiated power can be calculated using spherical coordinates as follows, where the surface area of the sphere is moved up into the numerator:

\[
D = \frac{A \cdot I(\theta, \phi)_{MAX}}{\int_{\phi=0}^{2\pi} \int_{\theta=0}^{2\pi} I(\theta, \phi)r^2 \sin \theta d\theta d\phi}
\]

(Equation 31.13)

Example 31.4: Determine the directivity of an infinitesimal dipole antenna.
In practice, Equation 31.13 is quite difficult to evaluate for anything more complex than an infinitesimal dipole or a small loop antenna. More typically, we calculate the directivity by measuring or observing the half-power beam width. The half-power beam width (HPBW) is just what it sounds like—the width of the beam for which the signal is at least half of its maximum power. This is shown graphically in Figure 31.4. We begin with a radiation pattern that represents the Poynting vector (power density) of an infinitesimal dipole antenna (duplicated from Figure 29.7). We then determine the peak intensity (labeled $S_{\text{MAX}}$), then divide this by two to find the points that correspond to half of the maximum power. Recall that the radius in this figure represents the intensity, so this can be done using a ruler. Once the two points at that power have been located, we just measure the angle between them. Here, it appears to be approximately 90°.

![Figure 31.4. Half-Power Beam Width of an Infinitesimal Dipole Antenna](image)

You can also determine the HPBW mathematically by determining the value of $\theta$ that corresponds to 50% of the maximum power.

Example 31.5: Determine the HPBW of an infinitesimal dipole antenna mathematically.
You must take special care to note whether the polar plot or function you are considering is a power plot (of the Poynting vector) or an amplitude plot (of the electric or magnetic field). If it is a power plot, you look for the points that are half of the peak, but if it is an amplitude plot, you look for the points that are 0.707 of the peak. (Recall that the Poynting vector will square this term, and \(0.707^2 = 0.5\).)

**Example 31.6:** Estimate the HPBW of a finite dipole antenna with \(L=\lambda\). The following image is a polar plot of electric field.

![Polar plot of electric field](image)

You can estimate the directivity of an antenna from its HPBW as follows:

\[
D \approx \frac{4\pi}{\Theta_{HP}\Phi_{HP}}
\]

(Equation 31.14)

\(\Theta_{HP}\) is the half-power beam width in the \(\theta\) direction (which we have been studying), while \(\Phi_{HP}\) is the half-power beam width in the \(\phi\) direction. Both of these values must be in radians. Typically for a dipole antenna, there is no \(\phi\) dependence of the radiation pattern, meaning that \(\Phi_{HP}\) is \(360^\circ\) or \(2\pi\) radians. For conical antennas like the Yagi, you can reasonably assume that the \(\phi\) dependence will match the \(\theta\) dependence, meaning that \(\Theta_{HP} = \Phi_{HP}\). It is important to note that this is an approximate calculation. Incidentally, the \(4\pi\) in the numerator comes from the fact that the surface area of a sphere is \(4\pi r^2\), and the \(r^2\) is canceled by a \(1/r^2\) in the Poynting vector.
Example 31.7: Given the answer to Example 31.5, estimate the directivity of an infinitesimal dipole antenna. Compare the result to the exact value from Example 31.4.

Example 31.8: Given the answer to Example 31.6, estimate the directivity of a finite dipole antenna with \( L = \lambda \).

31.3 Efficiency and Gain

Antennas are subject to physical losses, just like any other electrical device. These can take the form of leakage current through dielectrics and resistance losses in imperfect conductors. The efficiency of an antenna is defined as the radiated power divided by the power delivered to the antenna:

\[
\eta = \frac{P_{\text{rad}}}{P_{\text{in}}}
\]  
(Equation 31.15)

Thus, if the antenna has 95% efficiency and 100W is being delivered to the antenna, then 95W will be converted into electromagnetic radiation.

The gain of an antenna is a measure of how much of its input power is converted into a directed beam. The gain is calculated as the product of the efficiency and the directivity:

\[
G = \eta \cdot D
\]  
(Equation 31.16)

Since the efficiency of an antenna can take on any value between 0 and 1, and the directivity can take on any value greater than or equal to one, the gain can be any positive value.
**Example 31.9:** A dipole antenna with \( L = \lambda \) converts 96% of its input power to electromagnetic radiation. What is its gain?

### 31.4 Effective Area

The effective area of an antenna (also known as the aperture) is an important parameter for a receiving antenna. It is a measure of how effectively the antenna converts from electromagnetic radiation (measured by \( S_{AV} \)) to electrical power available at its output terminals \( (P_L) \). It is defined mathematically as:

\[
A_e = \frac{P_L}{S_{AV}}
\]

(Equation 31.17)

The effective area can be calculated from the directivity and wavelength as:

\[
A_e = \frac{\lambda^2}{4\pi} D
\]

(Equation 31.18)

Allowing for imperfect conductors and insulators in the antenna, we can rewrite Equation 31.18 to use gain (which includes efficiency) rather than directivity:

\[
A_e = \frac{\lambda^2}{4\pi} G
\]

(Equation 31.19)

**Example 31.10:** Given the answer to Example 31.8, what is the effective area of a finite dipole antenna with \( L = \lambda \)? Assume lossless insulators and conductors. The antenna is transmitting a signal at 10MHz.
31.5 Friis Transmission Equation

Ultimately, when we are using antennas, the goal is to get information from the input terminals of the transmitting antenna to the output terminals of the receiving antenna. Whatever happens in between, it is this transfer function that determines whether or not our wireless communication has been successful. The Friis transmission equation will allow us to determine this ratio. Figure 21.5 illustrates the antenna configuration that corresponds to the Friis transmission equation. The transmitting antenna, with gain $G_t$ and effective area $A_{e,t}$, is a distance $R$ from the receiving antenna, with gain $G_r$ and effective area $A_{e,r}$.

We know that the power radiated away from the transmitting antenna is the input power multiplied by the gain of the transmitting antenna in the direction of the receiver:

$$P_{rad} = P_t \cdot G_t$$  \hspace{1cm} (Equation 31.20)

This total power is spread out over a sphere of surface area $4\pi R^2$, so the Poynting vector, which is the power per unit area radiating from the transmitting antenna to the receiving antenna, can be written as:

$$S_{av} = \frac{P_t \cdot G_t}{4\pi R^2}$$  \hspace{1cm} (Equation 31.21)

The effective area of the receiving antenna can be used to convert $S_{av}$ to the power at the output terminals of the receiver:

$$P_r = S_{av} \cdot A_{e,r}$$  \hspace{1cm} (Equation 31.22)

Substituting Equation 31.21 into Equation 31.22 gives:
Solving Equation 31.19 for $G_t$ gives:

$$G_t = A_{e,t} \frac{4\pi}{\lambda^2} \quad \text{(Equation 31.23)}$$

Substituting this result into Equation 31.22 gives:

$$P_r = \frac{P_t \cdot A_{e,t} \frac{4\pi}{\lambda^2}}{4\pi R^2} \cdot A_{e,r} \quad \text{(Equation 31.24)}$$

Simplifying this expression and solving for the ratio of receiver power divided by transmitter power gives:

$$\frac{P_r}{P_t} = \frac{A_{e,t} \cdot A_{e,r}}{\lambda^2 R^2} \quad \text{(Equation 31.25)}$$

This is the Friis transmission equation. Notice the symmetry in this equation: If antenna A transmits to antenna B with a certain effectiveness, then antenna B will transmit to antenna A with the same effectiveness.

**Example 31.11:** Given the answers to previous examples in this chapter, if we apply 100W of a 10MHz signal to the input terminals of an antenna with $L=\lambda$ and receive the signal with an identical antenna 1km away, what will be the output power of the receiver? Assume both antennas are 95% efficient and that they are pointed directly toward each other, achieving maximum directivity.
31.6 Summary

- We can calculate the input impedance of an antenna as:
  \[ R_{in} = 2 \frac{P_{rad}}{I_0^2} \]

- The input impedance of an infinitesimal dipole antenna is:
  \[ R_{in} = 80\pi^2 \left( \frac{L}{\lambda} \right)^2 \]

- The input impedance of a finite dipole antenna can best be found by looking up the appropriate length in Figure 31.3.

- The directivity of an antenna is the maximum signal intensity in any direction divided by the average signal intensity. The directivity of an isotropic antenna is 1, and the directivity of an infinitesimal dipole antenna is 1.5.

- The half-power beam width (HPBW) is the angle over which the antenna is transmitting at least half of its maximum intensity. The HPBW for an infinitesimal dipole is 90°.

- The directivity can be approximated as follows:
  \[ D \approx \frac{4\pi}{\Theta_{HP} \Phi_{HP}} \]
  Here, \( \Theta_{HP} \) is the conventional HPBW, and \( \Phi_{HP} \) is the HPBW in the transverse direction. Typically, \( \Phi_{HP} \) is either \( 2\pi \) or is equal to \( \Theta_{HP} \).

- Efficiency measures how much of the power applied to an antenna is converted to radiation:
  \[ \eta = \frac{P_{rad}}{P_{in}} \]

- The gain of an antenna is the product of the efficiency and the directivity:
  \[ G = \eta \cdot D \]

- The effective area of an antenna measures how much of the incoming electromagnetic radiation is converted into electrical power at the output terminals:
  \[ A_e = \frac{P_L}{S_{av}} \]
• Effective area can be calculated as follows:

\[ A_e = \frac{\lambda^2}{4\pi} G \]

• The Friis transmission equation allows us to calculate the ratio of output power from the receiver divided by input power to the transmitter.

\[ \frac{P_r}{P_t} = \frac{A_{e,t} \cdot A_{e,r}}{\lambda^2 R^2} \]