Chapter 29: Electromagnetic Radiation and Infinitesimal Dipole Antennas

**Chapter Learning Objectives:** After completing this chapter the student will be able to:

1. Explain the physical phenomenon that leads to the radiation of electromagnetic waves.
2. Calculate the electric field and magnetic field generated in the far field of an infinitesimal dipole antenna.
3. Calculate the power radiating from an infinitesimal dipole antenna.
4. Explain the significance of a radiation pattern.

You can watch the video associated with this chapter at the following link:

**Historical Perspective:** Heinrich Hertz (1857-1894) was a German physicist who was the first to generate airborne electromagnetic waves. He was also the first to use both a dipole antenna and a loop antenna. The unit of frequency, Hertz, is named after him.

Photo credit: https://upload.wikimedia.org/wikipedia/commons/5/50/Heinrich_Rudolf_Hertz.jpg, [Public domain], via Wikimedia Commons.
29.1 Fundamentals of Electromagnetic Radiation

As we have already seen, static (motionless) electric charges produce electric fields and cause forces on other electric charges. We have also seen that charges moving at a constant velocity produce constant magnetic fields, and these magnetic fields will, in turn, create forces on other moving charges. It would be natural for you to take the next step and ask, “What happens when an electric charge accelerates?” The answer is that it creates an electromagnetic wave, which will propagate away from the accelerating charge at the speed of light. It would then be natural for you to ask, “Why?” This question will take a bit more effort to answer.

First, recall that the Poynting vector tells us that electromagnetic propagation will always be in the direction of $E \times H$:

$$S = E \times H$$  \hspace{1cm} (Copy of Equation 18.15)

This means that if we want to have electromagnetic propagation, there must be both an electric field and a magnetic field, and they must not be pointing in the same direction (which would lead to a cross-product of zero). In addition, since the radiation must be pointing away from the source, neither the electric field nor the magnetic field can be pointing radially away from the source. This is not a problem for $H$, which never points radially away from a charge, but it is a big problem for $E$, which (in our experience) always points radially away from a charge. We must find a way to create an electric field that has a transverse component in order to radiate energy away from the point source, as illustrated in Figure 29.1.

![Figure 29.1. Non-Radial Electric and Magnetic Fields Required for Electromagnetic Radiation](image)

We can also calculate the total power radiated away from a point source by integrating the Poynting vector over a closed surface:
We know that the radiated power will be pointing radially away from the point source, so it will be perpendicular to a sphere at every point. Thus, we only care about the magnitude of the electric field and magnetic field as well as the area of the sphere. Equation 29.1 can be simplified as follows:

\[ P_{rad} = |E| \cdot |H| \cdot A \]  

(Equation 29.2)

We can calculate the magnitude of the magnetic field as follows:

\[ |H| = \frac{|E|}{Z_0} \]  

(Equation 29.3)

Combining these two equations and substituting the surface area of a sphere, we find:

\[ P_{rad} = \frac{E^2}{Z_0} (4\pi r^2) \]  

(Equation 29.4)

However, we know that the total power radiated must be a constant (not increasing with the distance away from the point source). The only way this can happen is if E is inversely proportional to R:

\[ E \propto \frac{1}{r} \]  

(Equation 29.5)

So, we know that if we want to radiate electromagnetic fields, we must find electric fields that have a transverse component and vary inversely with r. Figure 29.2 starts our path to that discovery, showing the radial electric field lines that surround a static (motionless) charge.

Figure 29.2. Radial Electric Fields Surrounding a Static Charge
As we have already seen in other contexts, electric and magnetic fields propagate at the speed of light. This means that if the charge is suddenly accelerated for a brief period of time $\Delta t$, and it then maintains a constant velocity $v = a\Delta t$ after that brief acceleration, then the electric field lines that were emitted before the acceleration will continue to propagate along their original path, while the newer lines that were emitted during and after the acceleration will no longer line up with them. However, the electric field lines must be continuous, since the acceleration cannot be infinite. This situation is shown in Figure 29.3.

![Figure 29.3. Accelerating the Charge Introduces Diagonal Electric Field Lines](image)

Notice that the diagonal portions of the electric field lines (colored red in the electronic version) are required to connect the two sets of radial electric field lines, but the red portions themselves have both a radial and a transverse component. This is illustrated in Figure 29.4, which is a zoomed-in version of the top portion of Figure 29.3.

![Figure 29.4. Calculating the Transverse Component of the Diagonal Electric Field Line](image)
The blue line (labeled $E_t$) in Figure 29.4 is the portion that has the possibility of creating an electromagnetic wave. It is the horizontal component of the red line, which represents the complete electric field. In this figure, $\Delta t$ is the time that the particle was accelerated, and $t$ is the time it has moved at a constant velocity after acceleration. Thus, the horizontal side of the triangle has a length of $vt$, and the vertical leg has a length of $c\Delta t$, which is the distance that the electric field lines propagated while the charge was accelerating. We can write the following equation based on this triangle:

$$\frac{E_0}{E_t} = \frac{c\Delta t}{vt}$$  \hspace{1cm} (Equation 29.6)

We know that the radial component of the electric field, $E_0$, will obey the following equation:

$$E_0 = \frac{Q}{4\pi \epsilon_0 r^2}$$  \hspace{1cm} (Equation 29.7)

Substituting Equation 29.7 into Equation 29.6 and solving for $E_t$, we find:

$$E_t = \frac{Q}{4\pi \epsilon_0 r^2} \frac{vt}{c\Delta t}$$  \hspace{1cm} (Equation 29.8)

Substituting $r=ct$ for one power of $r$ in the denominator and rearranging the equation gives:

$$E_t = \frac{Q}{4\pi \epsilon_0 \Delta t c^2 t} \frac{v}{c} \frac{t}{r}$$  \hspace{1cm} (Equation 29.9)

Since $v/\Delta t=a$ and $\mu_0=1/(\epsilon_0 c^2)$, we can rearrange this to become:

$$E_t = \frac{Q}{4\pi r} \frac{a}{\mu_0}$$  \hspace{1cm} (Equation 29.10)

If you look carefully back at Figure 29.3, it turns out that the horizontal field lines (at 3:00 and 9:00), don’t have diagonal components. In fact, the size of the diagonal component varies with $\theta$, which is shown on Figure 29.3. The minimum diagonal component occurs at $\theta=0^\circ$ and $\theta=180^\circ$, while the maximum occurs at $\theta=90^\circ$ and $\theta=270^\circ$. This is satisfied by appending a $\sin(\theta)$ to the end of Equation 29.10, as shown:

$$E_t = \frac{Q}{4\pi} \frac{a}{r} \mu_0 \sin(\theta)$$  \hspace{1cm} (Equation 29.11)
Let’s take a moment to talk about Equation 29.11. First, it does have an inversely proportional relationship to $r$, just as we predicted back in Equation 29.5. It is also reasonable to assume that larger charges or higher levels of acceleration will lead to stronger electric fields, and that is also verified. It is also important to observe that the $\sin(\theta)$ factor indicates that the strongest electric field (and, therefore, the strongest electromagnetic radiation) will be at right angles to the direction of acceleration. Along the axis of acceleration, there will be no radiation emitted. This “directivity” of radiation will be an important consideration for us in the remainder of this book.

**Example 29.1:** A charged particle with $Q=0.1\, \text{C}$ is accelerated along positive $x$-axis at a rate of $a=10\, \text{m/s}^2$. What is the magnitude of the transverse electric field 2m away from the particle and at an angle of $45^\circ$?

We can next find the amplitude of the magnetic field at the same point by using the characteristic impedance of free space:

$$|H| = \frac{|E|}{Z_0}$$  \hspace{1cm} (Equation 29.12)

Substituting Equation 29.11 into this expression, we find:

$$H = \frac{Q}{4\pi Z_0 r} \frac{a}{\mu_0} \sin(\theta)$$  \hspace{1cm} (Equation 29.13)

Since we know that:

$$Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$  \hspace{1cm} (Copy of Equation 19.43)

We can rewrite Equation 29.13 as:

$$H = \frac{Q \sqrt{\varepsilon_0}}{4\pi \sqrt{\mu_0}} \frac{a}{r} \mu_0 \sin(\theta)$$  \hspace{1cm} (Equation 29.14)
Simplifying this equation gives:

\[ H = \frac{Q \sqrt{\varepsilon_0 \mu_0} a}{4\pi r} \sin(\theta) \]  \hspace{1cm} (Equation 29.15)

Since we also know:

\[ c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \]  \hspace{1cm} (Copy of Equation 18.43)

We can further simplify Equation 29.15 to:

\[ H = \frac{Q a}{4\pi c r} \sin(\theta) \]  \hspace{1cm} (Equation 29.16)

**Example 29.2:** What is the magnitude of the magnetic field that corresponds to the electric field at the same point in Example 29.1?

We know that the electric field from Equation 29.11 and the magnetic field from Equation 29.16 will cross to yield an electromagnetic wave pointing radially away from the accelerating charge. The power radiating with this wave can be calculated by the Poynting vector, which is calculated according to Equation 18.15:

\[ \mathbf{S} = \left[ \frac{Q a}{4\pi r} \mu_0 \sin(\theta) \right] \cdot \left[ \frac{Q a}{4\pi c r} \sin(\theta) \right] \mathbf{a}_r \]  \hspace{1cm} (Equation 29.17)

This can be simplified to:

\[ \mathbf{S} = \frac{Q^2 a^2}{16\pi^2 c r^2} \mu_0 \sin^2(\theta) \mathbf{a}_r \]  \hspace{1cm} (Equation 29.18)
Example 29.3: What is the Poynting vector that corresponds to Examples 29.1 and 29.2?

Finally, we can calculate the total radiated power from an accelerated charge. To do so, we must integrate $\mathbf{S}$ over a complete sphere at a radius $r$ away from the charge, as shown in Equation 29.1. We will use spherical coordinates and Equation 2.25 for the differential surface area:

$$P_{rad} = \int_0^{2\pi} \int_0^\pi \left( \frac{Q^2}{16\pi^2} \frac{a^2}{c} \mu_0 \sin^2 \theta \mathbf{a}_r \right) \cdot (r^2 \sin \theta d \phi d \theta d \mathbf{a}_r)$$  \hspace{1cm} \text{(Equation 29.19)}

Simplifying this equation and performing the dot product yields:

$$P_{rad} = \int_0^{2\pi} \int_0^\pi \frac{Q^2}{16\pi^2} \frac{a^2}{c} \mu_0 \sin^3 \theta d \phi d \theta$$  \hspace{1cm} \text{(Equation 29.20)}

Moving constants outside the integral and performing the trivial integral with respect to $\phi$ gives:

$$P_{rad} = 2\pi \frac{Q^2}{16\pi^2} \frac{a^2}{c} \mu_0 \int_0^\pi \sin^3 \theta d \theta$$  \hspace{1cm} \text{(Equation 29.21)}

The remaining definite integral can be evaluated to $4/3$:

$$P_{rad} = 2\pi \frac{Q^2}{16\pi^2} \frac{a^2}{c} \mu_0 \frac{4}{3}$$  \hspace{1cm} \text{(Equation 29.22)}

Simplifying this gives:

$$P_{rad} = \frac{Q^2 a^2}{6\pi} c \mu_0$$  \hspace{1cm} \text{(Equation 29.23)}

Finally, we know from Equation 18.43 that $\mu_0$ can be written as:

$$\mu_0 = \frac{1}{c^2 \epsilon_0}$$  \hspace{1cm} \text{(Equation 29.24)}

Substituting this expression into Equation 29.23 gives:
This is known as the Larmor Formula, and it demonstrates that the total amount of power radiated by an accelerating charge depends only on the magnitude of the charge, the rate of acceleration, and universal constants. Remember from Equation 29.18 that this radiation is not distributed uniformly—it is strongest in the direction perpendicular to the direction of acceleration and goes to zero along the axis of acceleration.

Example 29.4: What is the total power radiated by the accelerated charge of Example 29.1?

**Important Note:** Everything in the preceding derivation and, in fact, everything for the rest of this book will neglect any relativistic effects. This means that the charged particles will never be moving at anything approaching the speed of light, nor will the distances involved be so large that we need to consider any delays due to the speed of light. Including these effects makes these discussions much, much more complicated. If you’d like to know more about it, this is the field known as electrodynamics, and it has been the life’s work of several Nobel-prize-winning scientists over the past 125 years.

## 29.2 Infinitesimal Dipole Antenna

I hope you are now convinced that an accelerating charge will cause electromagnetic waves to propagate away from it. Of course, one way to accelerate a particle is along a straight line, but that would require a very long path, especially as the particle’s velocity increases. Two other possibilities are much more realistic:

1. Accelerating in one direction, then decelerating and reversing direction repeatedly. This is the principle behind a dipole antenna, and it is similar to the acceleration observed by a pendulum, which also constantly reverses direction.
2. Centripetal acceleration, which requires the particle to follow a circular path. This is the method used in a loop antenna, and it is very much like that associated with a merry-go-round or a satellite in orbit.
We will consider the dipole antenna first. In particular, we will consider an infinitesimal dipole, in which the length of the antenna is very, very short compared to the wavelength of the signal being applied to it:

\[ L << \lambda \]  

(Equation 29.26)

In practice, this means:

\[ L < \frac{\lambda}{50} \]  

(Equation 29.27)

An infinitesimal dipole antenna is illustrated in Figure 29.5.

![Infinitesimal Dipole Antenna](image)

\[ \text{Figure 29.5. Infinitesimal Dipole Antenna} \]

To solve for \( E \) and \( H \) (and \( S \)) created by this antenna, we must first begin by calculating the magnetic vector potential \( A \). As we saw in chapter 12, we can calculate \( A \) from \( J \) as follows:

\[
A(r) = \frac{\mu_0}{4\pi} \int_{\Delta v} \frac{J(r')}{r} dv' 
\]

(Copy of Equation 12.7)

Assuming that the current density is uniform throughout the cross-section of the antenna wire, we can show that:

\[
J(r) dv' = J(r) dS dl = I a_z dl'
\]

(Equation 29.28)

If we apply a time-harmonic signal to the antenna, the current can be written as:

\[
I = I_0 e^{-jk r}
\]

(Equation 29.29)

Substituting Equation 29.28 and 29.29 into 12.7, we can show that:

\[
A(r) = \frac{\mu_0}{4\pi} \int_{-L/2}^{+L/2} \frac{I_0 e^{-jk r}}{r} dl a_z
\]

(Equation 29.30)
If we assume that we are in the far field \((r \gg \lambda)\), then \(r\) does not vary with the position along the wire, meaning that the integral is simply integrating over a constant.

\[
A(r) = \frac{\mu_0 I_0 \mathcal{L}}{4\pi r} e^{-jkr} a_z
\]  
(Equation 29.31)

Rectangular coordinates were the easiest to use while we were focusing on the current flowing through the antenna, because the current flow was constrained to the \(z\) direction. But now we need to convert to spherical coordinates because the wave propagation will be strictly in a radial direction. To convert \(a_z\) into spherical coordinates, consider Figure 29.6.

\[\text{Figure 29.6. Converting } a_z \text{ into spherical coordinates.}\]

From this Figure, we can see that:

\[
A_r = A_z \cos \theta
\]  
(Equation 29.32)

\[
A_\theta = -A_z \sin \theta
\]  
(Equation 29.33)

We can then rewrite Equation 29.31 as:

\[
A(r) = \frac{\mu_0 I_0 \mathcal{L}}{4\pi r} e^{-jkr} (\cos \theta a_r - \sin \theta a_\theta)
\]  
(Equation 29.34)

Now that we have \(A(r)\) in spherical coordinates, we can calculate \(H(r)\):
In spherical coordinates, this can be calculated as follows:

\[ \mathbf{H}(\mathbf{r}) = \frac{1}{\mu_0 r} \left[ \frac{\partial(r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta} \right] \mathbf{a}_\phi \]  

(Equation 29.35)

This results in:

\[ \mathbf{H}(\mathbf{r}) = -\frac{I_0}{4\pi} k^2 \sin \theta \left[ \frac{1}{jkr} - \frac{1}{(jkr)^2} \right] e^{-jk r} \mathbf{a}_\phi \]  

(Equation 29.36)

Neglecting the second term because \( r \) is very large, we can simplify this to:

\[ \mathbf{H}(\mathbf{r}) \approx \frac{jk I_0 \mathcal{L}}{4\pi r} e^{-jk r} \sin \theta \mathbf{a}_\phi \]  

(Equation 29.37)

**Example 29.5:** An infinitesimal dipole antenna with length 1cm has a sinusoidal current of amplitude 1A and frequency 1MHz applied to it. What is the amplitude of the magnetic field at a distance of 5m and at an angle of 30° away from the direction of the antenna?

We can now find the electric field that corresponds to the magnetic field:

\[ \mathbf{E}(\mathbf{r}) = \frac{1}{j \omega \varepsilon_0} \nabla \times \mathbf{H}(\mathbf{r}) \]  

(Equation 29.38)

In spherical coordinates, this can be calculated as follows:

\[ \mathbf{E}(\mathbf{r}) = \frac{1}{j \omega \varepsilon_0} \left[ \frac{1}{r \sin \theta} \frac{\partial [\sin \theta H_\phi]}{\partial \theta} \mathbf{a}_r - \frac{1}{r} \frac{\partial (r H_\phi)}{\partial r} \mathbf{a}_\theta \right] \]  

(Equation 29.39)
This results in the following expression for electric field in the far field:

\[
E(r) \approx \frac{jZ_0kI_0L e^{-jkr}}{4\pi r} \sin \theta a_\theta
\]  
(Equation 29.40)

\[\text{Example 29.6:}\] What is the electric field that corresponds to the magnetic field from Example 29.5?

Finally, we will calculate the average Poynting vector for this antenna. Remember that the Poynting vector will vary with time in this case, but we can calculate the average energy flow as shown in Chapter 18:

\[
S_{av}(r) = \frac{1}{2} \text{Re} \left[ E(r) \times H^*(r) \right]
\]  
(Copy of Equation 18.18)

Substituting Equations 29.37 and 29.40 into Equation 18.18, we find:

\[
S_{av}(r) = \frac{1}{2} \text{Re} \left[ \frac{jZ_0kI_0L e^{-jkr}}{4\pi r} \sin \theta a_\theta \times \frac{-j k I_0 L e^{+jkr}}{4\pi r} \sin \theta a_\phi \right]
\]  
(Equation 29.41)

Combining terms and performing the cross product yields:

\[
S_{av}(r) = \frac{1}{2} Z_0 \left( \frac{k I_0 L}{16 \pi^2} \right)^2 \frac{1}{r^2} \sin^2 \theta a_r
\]  
(Equation 29.42)

\[\text{Example 29.7:}\] What is the average Poynting vector that corresponds to Examples 29.5 and 29.6?
We can plot the amplitude of the average Poynting vector in a polar plot, as shown in Figure 29.7. A polar plot shows amplitude as a function of the angle $\theta$. In this figure, we can see that the maximum power transmitted is in the direction perpendicular to the antenna, while there is no power being transmitted parallel to the antenna. This corresponds very well with our observations of the directionality of power transmitted by a simple accelerating charge. Note that the regions of maximum power transmission are referred to as “lobes,” while regions of minimum or zero power transmission are referred to as “nulls.”

![Figure 29.7. Polar Plot of Average Poynting Vector for Infinitesimal Dipole](image)

Finally, we can calculate the total power radiated by the antenna if we integrate the average Poynting vector over a closed surface (a sphere, in this case).

$$P_{rad} = \int_{\Delta S} S_{av} \bullet dS$$  \hspace{1cm} (Equation 29.43)

Using the differential surface area in spherical coordinates, we can rewrite this as:

$$P_{rad} = \int_{\Delta S} \frac{1}{2} Z_0 \left(\frac{k I_0 L}{\omega c^2}\right)^2 \frac{1}{r^2} \sin^2 \theta \mathbf{a}_r \bullet r^2 \sin \theta d\theta d\phi \mathbf{a}_r$$  \hspace{1cm} (Equation 29.44)

This can be simplified as follows:
\[ P_{rad} = \frac{1}{2} Z_0 \left( \frac{(kI_0L)^2}{16\pi^2} \right) \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin^3 \theta d\theta d\phi \]  
(Equation 29.45)

The integral with respect to \( \phi \) is simply \( 2\pi \), while the integral with respect to \( \theta \) is \( \frac{4}{3} \).

\[ P_{rad} = \frac{1}{2} Z_0 \left( \frac{(kI_0L)^2}{16\pi^2} \right) 2\pi \frac{4}{3} \]  
(Equation 29.46)

This can be simplified as follows:

\[ P_{rad} = Z_0 \left( \frac{(kI_0L)^2}{12\pi} \right) \]  
(Equation 29.47)

Example 29.8: What is the total radiated power for the antenna from Examples 29.5-29.7?

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29.3 Summary

- Electromagnetic radiation is created by accelerating electric charges. This acceleration creates a transverse electric field and a corresponding magnetic field:

\[ E_t = \frac{Q}{4\pi r} \alpha \mu_0 \sin(\theta) \quad H = \frac{Q}{4\pi cr} \sin(\theta) \]

- When combined, this electric field and magnetic field create an electromagnetic wave. The Poynting vector corresponding to this wave is:
This demonstrates that the waves tend to be generated perpendicular to the direction of acceleration. The total radiated power can be calculated from the Larmor Formula:

\[
P_{\text{rad}} = \frac{Q^2 a^2}{6\pi \epsilon_0 c^3}
\]

Infinitesimal dipole antennas are much shorter than the wavelength they are transmitting:

\[
\mathcal{L} < \frac{\lambda}{50}
\]

The magnetic and electric field created by an infinitesimal dipole are:

\[
\mathbf{H}(r) \approx \frac{jkI_0 \mathcal{L} e^{-jkr}}{4\pi} \frac{\sin \theta}{r} \mathbf{a}_\phi \quad \mathbf{E}(r) \approx \frac{jZ_0 kI_0 \mathcal{L} e^{-jkr}}{4\pi} \frac{\sin \theta}{r} \mathbf{a}_\theta
\]

The corresponding average Poynting vector is:

\[
\mathbf{S}_{\text{av}}(r) = \frac{1}{2} Z_0 \left( \frac{kI_0 \mathcal{L}}{16\pi^2} \right)^2 \frac{1}{r^2} \sin^2 \theta \mathbf{a}_r
\]

The total radiated power is:

\[
P_{\text{rad}} = Z_0 \left( \frac{kI_0 \mathcal{L}}{12\pi} \right)^2
\]