Chapter 24: Terminations and Reflections

Chapter Learning Objectives: After completing this chapter the student will be able to:

- Calculate the characteristic impedance and propagation velocity for different types of transmission lines.
- Calculate the reflection coefficient for the termination of a transmission line.
- Calculate the Voltage Standing Wave Ratio (VSWR) for the termination of a transmission line.

You can watch the video associated with this chapter at the following link: VIDEO

Historical Perspective: James West (1931-present) is an American inventor who co-invented the foil electret microphone in 1962. Nearly 90% of today’s microphones use a version of his invention. Dr. West spent nearly 40 years at Bell Laboratories, and he has more than 40 U.S. patents.

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24.1 Sinusoidal Waves and Characteristic Impedance

In the previous chapter, we found that a transmission line with a capacitance and inductance per unit meter will obey the wave equation for both voltage and current, which means that waves of all types will propagate along them at a velocity specified by:

\[ v = \frac{1}{\sqrt{\hat{L}\hat{C}}} \quad \text{(Copy of Equation 23.18)} \]

This result is not specific to time-harmonic (sinusoidal) waves, but sinusoidal waves are the most important example of waves on a transmission line. Focusing first on the voltage, we know that if it is time-harmonic, then we can separate the time dependence into a complex exponential term:

\[ V(z, t) = \text{Re}\left[V(z)e^{j\omega t}\right] \quad \text{(Equation 24.1)} \]

The spatial relationship \( V(z) \) is not necessarily sinusoidal yet, so we will leave it in the general form \( V(z) \).

Recalling the wave equation for voltage from the previous chapter:

\[ \frac{\partial^2 V(z, t)}{\partial z^2} - \hat{L}\hat{C}\frac{\partial^2 V(z, t)}{\partial t^2} = 0 \quad \text{(Copy of Equation 23.16)} \]

We will now substitute Equation 24.1 in for \( V(z,t) \):

\[ \frac{\partial^2 (V(z)e^{j\omega t})}{\partial^2 z} - \hat{L}\hat{C}\frac{\partial^2 (V(z)e^{j\omega t})}{\partial^2 t} = 0 \quad \text{(Equation 24.2)} \]

Since the first term is a partial derivative with respect to \( z \), we can pull the exponent out. Similarly, the \( V(z) \) is unaffected by the partial derivative with respect to \( t \), and the second derivative of the exponent introduces two \( j\omega \) factors:

\[ \frac{\partial^2 (V(z))}{\partial^2 z} e^{j\omega t} - \hat{L}\hat{C}(j\omega)^2 V(z)e^{j\omega t} = 0 \quad \text{(Equation 24.3)} \]

Cancelling the exponent terms and using \( j^2 = -1 \) gives:

\[ \frac{\partial^2 (V(z))}{\partial^2 z} + \hat{L}\hat{C}\omega^2 V(z) = 0 \quad \text{(Equation 24.4)} \]
This is the form of the one-dimensional Helmholtz equation, as we saw for the propagation of electromagnetic waves in free space:

\[
\frac{d^2V(z)}{dz^2} + k^2V(z) = 0 \quad \text{(Equation 24.5)}
\]

Where the constant \( k \) is defined to be:

\[
k = \omega \sqrt{\hat{L} \hat{C}} = \frac{\omega}{v} \quad \text{(Equation 24.6)}
\]

The solution to Equation 24.5 has the general form:

\[
V(z) = Ae^{-jkz} + Be^{jkz} \quad \text{(Equation 24.7)}
\]

This, of course, represents a sinusoidal wave in space, with a component moving toward the right and a component moving toward the left. So, just as we saw before, if we assume that the wave is time-harmonic, then the only way that that can be a solution to the wave equation is if Equation 24.6 is true and if the wave also varies sinusoidally in space.

If we also assume that the current is time-harmonic (which it must be, since voltage and current must stay in phase with each other in order for the wave to propagate):

\[
I(z, t) = Re\left[I(z)e^{j\omega t}\right] \quad \text{(Equation 24.8)}
\]

A similar analysis of the current will yield a corresponding differential equation:

\[
\frac{d^2I(z)}{dz^2} + k^2I(z) = 0 \quad \text{(Equation 24.9)}
\]

This equation will have the same general solution, but with different constants:

\[
I(z) = Ce^{-jkz} + De^{jkz} \quad \text{(Equation 24.10)}
\]

For now, let’s assume that there is only a positive-moving component to the wave. Combining Equation 24.1 with the positive-moving term in Equation 24.7 gives:

\[
V(z, t) = V_0e^{j(\omega t - kz)} \quad \text{(Equation 24.11)}
\]

Here, we have renamed the constant \( A \) to be \( V_0 \) since there is only one term. Similarly for current:
\[ I(z, t) = I_0 e^{j(\omega t - kz)} \]  
(Equation 24.12)

Now, we can substitute these two expressions into either of the transmission line equations from the previous chapter. We will use Equation 23.13.

\[
\frac{\partial V(z, t)}{\partial z} = -\hat{L} \frac{\partial I(z, t)}{\partial t} \quad \text{(Copy of Equation 23.13)}
\]

\[
\frac{\partial (V_0 e^{j(\omega t - kz)})}{\partial z} = -\hat{L} \frac{\partial (I_0 e^{j(\omega t - kz)})}{\partial t} \quad \text{(Equation 24.13)}
\]

Performing the indicated partial derivatives gives:

\[
(-jk)V_0 e^{j(\omega t - kz)} = -\hat{L}(j\omega)I_0 e^{j(\omega t - kz)} \quad \text{(Equation 24.14)}
\]

Simplifying this equation yields:

\[
kV(z) = \omega \hat{L} I(z) \quad \text{(Equation 24.15)}
\]

Solving this equation for \( I(z) \) gives:

\[
I(z) = \frac{k}{\omega \hat{L}} V(z) \quad \text{(Equation 24.16)}
\]

Defining the characteristic impedance of the transmission line to be the voltage divided by the current, we find:

\[
Z_c = \frac{V(z)}{I(z)} = \frac{\omega \hat{L}}{k} \quad \text{(Equation 24.17)}
\]

Substituting Equation 24.6 into this equation yields the final form of the characteristic impedance:

\[
Z_c = \sqrt{\frac{\hat{L}}{\hat{C}}} \quad \text{(Equation 24.18)}
\]
So, if the voltage along the transmission line is time-harmonic, then it will also vary sinusoidally in space. Furthermore, the current will vary in precisely the same manner, and the ration between the magnitude of the voltage and that of the current will be constant and can be easily calculated.

Of course, you can calculate the capacitance and inductance per unit length and then use Equation 24.18 to calculate the characteristic impedance, but the following table cuts this to one step by substituting Equations 23.19-23.24 into Equations 23.18 and 24.18.

<table>
<thead>
<tr>
<th>Table 24.1 Velocity and Impedance for Transmission Lines</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Velocity</strong></td>
</tr>
<tr>
<td>------------------</td>
</tr>
<tr>
<td>Coaxial Cable</td>
</tr>
<tr>
<td>Microstrip Line</td>
</tr>
<tr>
<td>Twin Lead</td>
</tr>
</tbody>
</table>

Using this table to calculate $v$ and $Z_c$ will eliminate the need to calculate capacitance and impedance as intermediate steps. Notice that the propagation velocity of all three types of transmission lines is the same, and it only depends on the relative permittivity and relative permeability of the material between the conductors. Since $mr$ is almost always equal to one in these transmission lines, the velocity can be quickly and easily calculated as follows:

$$v = \frac{C}{\sqrt{\epsilon_T}}$$

(Equation 24.19)

**Example 24.1:** Determine the propagation velocity and characteristic impedance of a coaxial cable with an inner radius of 0.5cm, an outer radius of 1cm, and a relative dielectric constant of 10.
**Example 24.2:** Determine the propagation velocity and characteristic impedance of a twin lead transmission line in which each wire has a radius of 0.1 mm, the distance between the wires is 2 mm, and the material between them has a relative dielectric constant of 5.

### 24.2 Terminations and Reflection Coefficient

In the previous section, we made the assumption that only a positive-moving wave is present on the transmission line. This is true if the wave is generated at negative infinity and has never encountered any discontinuity or load during its travels. But every transmission line must terminate (end), and when that happens, it is often the case that a signal will be reflected from the termination and will begin to move in the opposite direction.

We have already seen an equation for the voltage of a positive- and negative-moving wave on the same line:

\[ V(z) = A e^{-j k z} + B e^{j k z} \]  
(Copy of Equation 24.7)

We know that the magnitude of the positive-moving current wave will be \(1/Z_c\) as large as the positive-moving voltage wave. If a similar derivation is performed on negative-moving waves, we would find that the amplitude of the current wave is also negative. This negative sign appears in the derivation at the point of Equation 24.14 because no negative sign is introduced on the left side. This means that the current will have this general form:

\[ I(z) = \frac{1}{Z_c}(A e^{-j k z} - B e^{j k z}) \]  
(Equation 24.20)

Consider the situation where these two equations describe the voltage and current on a wire with characteristic impedance \(Z_c\) that is terminated by a load with impedance \(Z_L\), as shown in Figure 24.1.

![Figure 24.1. Transmission Line with Termination](image)
Notice that, although \( z \) is increasing toward the right, we have selected the terminating end of the line to be \( z=0 \). This means that the rest of the wire will exist in the negative range of \( z \).

Due to the interactions among the positive-moving waves and the negative-moving waves, the impedance will vary from point to point along the line. (We will later discover a tool that will help us to deal with this complexity.) The impedance, which is the voltage at each point divided by the current at that point, can be written as:

\[
Z(z) = \frac{V(z)}{I(z)} = \frac{Ae^{-jkz} + Be^{jkz}}{\frac{1}{Z_c}(Ae^{-jkz} - Be^{jkz})} \tag{Equation 24.21}
\]

If we evaluate this equation at \( z=0 \) (the termination point), then this ratio must be equal to the load impedance \( Z_L \), because the voltage and current at the end of the wire must be the same as the voltage and current associated with the load impedance:

\[
Z(0) = \frac{V(0)}{I(0)} = \frac{Ae^{-jk0} + Be^{jk0}}{\frac{1}{Z_c}(Ae^{-jk0} - Be^{jk0})} = Z_L \tag{Equation 24.22}
\]

Evaluating each of the exponential terms with \( e^0 = 1 \), we obtain:

\[
Z_L = Z_c \frac{A + B}{A - B} \tag{Equation 24.23}
\]

Next, we will define the reflection coefficient \( \Gamma \) to be the magnitude of the reflected (negative-moving) wave divided by the magnitude of the incoming (positive-moving) wave:

\[
\Gamma \equiv \frac{B}{A} \tag{Equation 24.24}
\]

We can rearrange Equation 24.23 to obtain an expression for the reflection coefficient:

\[
\Gamma = \frac{Z_L - Z_C}{Z_L + Z_C} \tag{Equation 24.25}
\]

From this equation, we can see that the proportion of an incoming wave that is reflected at the termination is only a function of the characteristic impedance of the transmission line and the impedance of the load attached to it.
Example 24.3: Derive Equation 24.25 from Equation 24.23 and the definition of $\Gamma$.

Example 24.4: Determine the reflection coefficient when the transmission line of Example 24.1 is terminated in a $100\Omega$ load.

Example 24.5: A coaxial cable has a discontinuity in the dielectric constant at $z=0$. For $z<0$, $\varepsilon_r=4$, and for $z>0$, $\varepsilon_r=6$. All other parameters of the cable stay the same. Calculate the reflection coefficient.

We can express the total voltage and current (the sum of the incident and reflected waves) at any point along the wire as follows:

\[
V(z) = V_0 \left( e^{-jkz} + \Gamma e^{jkz} \right) \quad \text{(Equation 24.26)}
\]

\[
I(z) = \frac{V_0}{Z_c} \left( e^{-jkz} - \Gamma e^{jkz} \right) \quad \text{(Equation 24.27)}
\]

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Be sure to remember that the reflected current term has a negative sign. Omitting this negative sign will yield incorrect results in the upcoming analyses.

There are three special cases of load impedance that deserve extra attention. The first is a short circuit, in which $Z_L=0$. This is illustrated in Figure 24.2.

![Figure 24.2. Transmission Line Terminated with a Short Circuit](image)

When we substitute $Z_L=0$ in Equation 24.25, we find:

$$\Gamma = \frac{Z_L - Z_C}{Z_L + Z_C} = \frac{0 - Z_C}{0 + Z_C} = -1 \quad \text{(Equation 24.28)}$$

So, when the cable is short-circuited, the reflection coefficient is -1, which means that the total voltage (from Equation 24.26 and Equation 24.1) is:

$$V(z, t) = Re \left[ V_0 \left( e^{-jkz} - e^{jkz} \right) e^{j\omega t} \right] \quad \text{(Equation 24.29)}$$

This simplifies as follows to a standing wave, just as we saw when an electromagnetic wave reflected off of a perfect conductor:

$$V(z, t) = -2V_0 \sin(kz) \sin(\omega t) \quad \text{(Equation 24.30)}$$

Equation 24.30 is illustrated below in Figure 24.3. Notice that the right end of the wire has a voltage of zero, which is necessary because it is short-circuited.

![Figure 24.3. Voltage Along a Short-Circuit-Terminated Transmission Line](image)
Next, we will consider an open circuit, in which the load impedance is infinitely large. This is shown in Figure 24.4:

\[ Z_L = \infty \]

\[ Z_C \]

\[ z = 0 \]

**Figure 24.4. Transmission Line Terminated with an Open Circuit**

If we substitute an infinite load impedance into Equation 24.25, we find that the reflection coefficient is +1:

\[ \Gamma = \frac{Z_L - Z_C}{Z_L + Z_C} = \frac{\infty - Z_C}{\infty + Z_C} = +1 \]  

(Equation 24.31)

This reflection coefficient yields a total voltage expression of:

\[ V(z, t) = Re \left[ V_0 \left( e^{-jkz} + e^{jkz} \right) e^{j\omega t} \right] \]  

(Equation 24.32)

This expression simplifies to:

\[ V(z, t) = 2V_0 \cos(kz) \sin(\omega t) \]  

(Equation 24.33)

This expression is illustrated in Figure 24.5. Notice that now the voltage is not zero at the termination point, since it is an open circuit. A similar plot of current would show zero current at the open-circuit termination.

**Figure 24.5. Voltage Along an Open-Circuit-Terminated Transmission Line**
The third special case is when the load impedance is equal to the characteristic impedance of the load. This is known as an impedance-matched line, and achieving impedance matching will be the subject of chapter 25. In general, we always want our lines to be impedance matched if possible, because reflections are problematic in most systems. Notice that when the system is impedance matched, the reflection coefficient is zero:

\[
\Gamma = \frac{Z_L - Z_C}{Z_L + Z_C} = \frac{Z_C - Z_C}{Z_C + Z_C} = 0 \quad \text{(Equation 24.34)}
\]

### 24.3 Voltage Standing Wave Ratio (VSWR)

When a line is terminated with either a short circuit or an open circuit, it will create standing waves. When it is impedance matched, there will be no reflections, and only the positive-moving wave will be present on the line. But most of the time, we will not observe one of those three special cases. In the general case, the total voltage will be a sine wave that moves toward the load while also changing amplitude between a minimum value and a maximum value. Recall the equation for the total voltage:

\[
V(z) = V_0\left(e^{-j kz} + \Gamma e^{j kz}\right) \quad \text{(Copy of Equation 24.26)}
\]

The maximum of this wave will occur when the incident wave and the reflected wave are perfectly in phase with each other. In that case, the reflected wave amplitude of \( V_0\Gamma \) will add to the incident wave amplitude of \( V_0 \):

\[
V_{\text{max}} = V_0\left(1 + |\Gamma|\right) \quad \text{(Equation 24.35)}
\]

Similarly, the minimum value will occur when the incident and reflected waves are 180° out of phase. When that happens, the amplitude of the reflected wave will be subtracted from the amplitude of the incident wave:

\[
V_{\text{min}} = V_0\left(1 - |\Gamma|\right) \quad \text{(Equation 24.36)}
\]

We will define a quantity called the Voltage Standing Wave Ratio (VSWR), which is the ratio of the maximum voltage divided by the minimum voltage:

\[
VSWR \equiv \frac{V_{\text{max}}}{V_{\text{min}}} \quad \text{(Equation 24.37)}
\]
Substituting Equations 24.35 and 24.36 into Equation 24.37 gives:

\[
VSWR = \frac{1 + |\Gamma|}{1 - |\Gamma|}
\]  \(\text{(Equation 24.38)}\)

We can rearrange this equation to solve for the magnitude of the reflection coefficient in terms of the VSWR:

\[
|\Gamma| = \frac{VSWR - 1}{VSWR + 1}
\]  \(\text{(Equation 24.39)}\)

If the line is impedance matched, then the reflection coefficient will be zero, and the VSWR will be one. VSWR=1 is the ideal case, and so any VSWR substantially higher than one indicates that the line is poorly matched to the load, which indicates that a large amount of undesirable reflections will occur.

Both short-circuited lines and open-circuited lines will have an infinite VSWR. This is the worst possible case.

**Example 24.6**: Calculate the VSWR for the transmission line discontinuity in Example 24.5.

### 24.4 Summary

- When a sinusoidal wave flows down a transmission line, it will be sinusoidal in both time and space. The voltage and current waves will move at the same velocity and will remain in phase with one another. The characteristic impedance of the line, which is the amplitude of the positive-moving voltage divided by the amplitude of the positive-moving current, is:

\[
Z_c = \sqrt{\frac{L}{C}}
\]
• The velocity of waves along a transmission line depends only on the relative dielectric constant of the material between the conductors:

\[ v = \frac{c}{\sqrt{\varepsilon_r}} \]

• The characteristic impedance of each of the three major types of transmission lines can be calculated as follows:

<table>
<thead>
<tr>
<th>Transmission Line</th>
<th>Characteristic Impedance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coaxial Cable</td>
<td>( Z_c = \sqrt{\frac{\mu}{\varepsilon}} \left( \frac{\ln(b/a)}{2\pi} \right) )</td>
</tr>
<tr>
<td>Microstrip Line</td>
<td>( Z_c = \sqrt{\frac{\mu}{\varepsilon}} \left( \frac{d}{w} \right) )</td>
</tr>
<tr>
<td>Twin Lead</td>
<td>( Z_c = \sqrt{\frac{\mu}{\varepsilon}} \left( \frac{\cosh^{-1}(D/a)}{\pi} \right) )</td>
</tr>
</tbody>
</table>

• When a transmission line is terminated in a load impedance, reflections can occur. The reflection coefficient can be calculated as follows:

\[ \Gamma = \frac{Z_L - Z_C}{Z_L + Z_C} \]

• For a short-circuited line, \( \Gamma = -1 \) and VSWR=\( \infty \). For an open-circuited line, \( \Gamma = +1 \) and VSWR=\( \infty \). For an impedance-matched line, \( \Gamma = 0 \) and VSWR=1.

• The VSWR is the ratio between the maximum voltage and the minimum voltage. It can be calculated as follows:

\[ VSWR = \frac{1 + |\Gamma|}{1 - |\Gamma|} \]

• If you know the VSWR, you can find the reflection coefficient as follows:

\[ |\Gamma| = \frac{VSWR - 1}{VSWR + 1} \]