Chapter Learning Objectives: After completing this chapter the student will be able to:

- Express a vector between two points in rectangular (Cartesian), cylindrical, or spherical coordinates.
- Transform a vector between any pair of the three coordinate systems.
- Determine a differential length, differential surface area, and differential volume in all three coordinate systems.

You can watch the video associated with this chapter at the following link:

Historical Perspective: Rectangular coordinates are sometimes referred to as “Cartesian” coordinates in honor of philosopher and mathematician René Descartes, the 17th century mathematician who founded analytical geometry.

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2.1 Coordinate Systems

As we saw in the previous chapter, a vector in three-dimensional space can be defined as a linear superposition of three orthogonal (perpendicular) unit vectors. Typically, we use a “coordinate system” to automatically determine the three orthogonal unit vectors we need. There are many valid coordinate systems, but we will study three: rectangular (Cartesian), cylindrical, and spherical. Each of these coordinate systems will play an important part in solving problems in this class.

2.2 The Rectangular Coordinate System

We used the rectangular coordinate system in the previous lesson without even really thinking about it. As we saw then, the three orthogonal unit vectors are \( \mathbf{a}_x, \mathbf{a}_y, \) and \( \mathbf{a}_z, \) each of which points in the direction of the corresponding axis. These three vectors, multiplied by corresponding constants, can be added together to yield any vector in three-dimensional space. An example is shown in Figure 2.1.

\[
\mathbf{A} = 2\hat{a}_x + 4\hat{a}_y + 3\hat{a}_z
\]

Figure 2.1. Representing a vector in rectangular coordinates.

Another thing we have been assuming without talking about it is the use of a right-handed coordinate system. In short, there are two ways that coordinate systems can be designed: right-handed and left-handed. As far as I can tell, everyone use the right-handed coordinate system, which is defined by the following relationships:

\[
\mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z \quad \mathbf{a}_y \times \mathbf{a}_z = \mathbf{a}_x \quad \mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y
\]

(Equation 2.1)
In practice, it is sufficient to verify that one of these equations is true, because then the other two are guaranteed to also be true. Every figure I share with you in this book will use a right-handed coordinate system.

Many times in this book, we will need to work with differential quantities. Think of a differential quantity as one that is so small that you can assume it is infinitely small. We will work with differential volume, surface area, and length. In rectangular coordinates, you can think of a differential element as looking like a small cube, as shown in Figure 2.2.

![Figure 2.2. A differential element in rectangular coordinates.](image)

Although this cube is shown as somewhat large, this is just so we can effectively label it. Remember that in reality, it is infinitesimally small. You can see that the three edges are labeled $dx$, $dy$, and $dz$, and that makes it easy to calculate the differential volume:

$$dv = dx \cdot dy \cdot dz$$  \hspace{1cm} (Equation 2.2)

In vector calculus, “areas” are defined to be vectors that are pointing perpendicular to the surface and have a magnitude equal to the area of the surface. For a closed surface (like a cube), the vector always points outward as shown in Figure 2.2. Notice that for a cube, there will be six sides, which means six differential surface areas and six vectors. Three of them are visible and labeled in Figure 2.2. The equations for all six surfaces are:

$$ds_x = \pm dy \cdot dz \cdot a_x$$  \hspace{1cm} (Equation 2.3)

$$ds_y = \pm dx \cdot dz \cdot a_y$$  \hspace{1cm} (Equation 2.4)

$$ds_z = \pm dx \cdot dy \cdot a_z$$  \hspace{1cm} (Equation 2.5)

Differential length is the distance from the corner nearest the origin to the corner furthest from the origin. Referring back to Figure 2.2, we can see that this can be represented as:
\[ dl = dx \cdot a_x + dy \cdot a_y + dz \cdot a_z \] (Equation 2.6)

2.3 The Cylindrical Coordinate System

The cylindrical coordinate system is the second of the three major coordinate systems. To locate a point in cylindrical coordinates, you must specify one angle \( \phi \) and two distances \( A_\rho \) and \( A_z \). You first point in the direction of the angle, then move outward as specified by \( A_\rho \) and then upward as specified by \( A_z \). This is illustrated in Figure 2.3.

![Figure 2.3. Representing a vector in cylindrical coordinates.]

One thing to notice about cylindrical coordinates is that, although there is a unit vector \( a_\phi \), we don’t really “move” in the direction of \( a_\phi \). It is strictly an “aiming” operation, preparing to move first in the \( \rho \) direction outward and then in the \( z \) direction upward. For this reason, we always append “with \( \phi \)” and its value at the end of the vector.

It’s also worth noting that \( a_z \) in cylindrical coordinates is still in the direction of the \( z \)-axis, which means that \( a_z \) in cylindrical coordinates is precisely the same \( a_z \) as in rectangular coordinates.

We can once again identify three cross product identities that will be true in cylindrical coordinates for a right-handed coordinate system:

\[
\begin{align*}
  a_\rho \times a_\phi &= a_z \\
  a_\phi \times a_z &= a_\rho \\
  a_z \times a_\rho &= a_\phi 
\end{align*}
\] (Equation 2.7)
We will once again need to study a differential element in order to determine the differential volume, differential surface areas, and differential length in cylindrical coordinates, as shown in Figure 2.4.

![Figure 2.4. A differential element in cylindrical coordinates.](image)

Of special interest in Figure 2.4 is the length of the inner curved surface, which is $\rho d\phi$. Remember that this side needs to be a length, and $d\phi$ by itself is not a length, it is a differential angle. (This differential angle is shown in the x-y plane at the bottom of the figure.) To make $d\phi$ into an arc length, we must multiply it by the length of the radius, which is $\rho$ in this case. Thus, $\rho d\phi$ is the arc length of the inner surface.

Although the outer surface (the one furthest from the z-axis) looks like it is longer than the inner surface (the one closest to the z-axis), this is an artifact of how the differential element is drawn. Remember that $d\rho$ is actually infinitesimally small, and so those two surfaces are actually infinitesimally close to each other, meaning they are the same size. Although it doesn’t look like it in Figure 2.4, this differential element is still a cube.

Thus, considering Figure 2.4, we find the following differential volume in cylindrical coordinates:

$$dv = \rho d\phi \cdot d\rho \cdot dz = \rho \cdot d\rho \cdot d\phi \cdot dz$$  \hspace{1cm} (Equation 2.8)

The differential surface areas (all six of them) can be represented by the following equations:

$$ds_{\rho} = \pm \rho \cdot d\phi \cdot dz \cdot a_{\rho}$$  \hspace{1cm} (Equation 2.9)

$$ds_{\phi} = \pm d\rho \cdot dz \cdot a_{\phi}$$  \hspace{1cm} (Equation 2.10)

$$ds_{z} = \pm \rho \cdot d\rho \cdot d\phi \cdot a_{z}$$  \hspace{1cm} (Equation 2.11)
Finally, the differential length between opposite corners of the differential element can be written as:

\[ dl = d\rho \cdot a_\rho + \rho \cdot d\phi \cdot a_\phi + dz \cdot a_z \] (Equation 2.12)

### 2.4 Converting Vectors Between Rectangular and Cylindrical Systems

Since both rectangular and cylindrical coordinate systems can be used to represent any point in three dimensional space, it follows that there must be a mapping or conversion from one system to the other.

To convert a vector, let’s first consider what the two vectors will look like:

\[ \mathbf{A} = A_x \mathbf{a}_x + A_y \mathbf{a}_y + A_z \mathbf{a}_z \] (Equation 2.13)

\[ \mathbf{A} = A_\rho \mathbf{u}_\rho + A_z \mathbf{u}_z \] with \( \phi \) (Equation 2.14)

Note that, although there is no \( A_\phi \) component in Equation 2.14, we must calculate and report \( \phi \) as part of the final answer.

To perform the conversion from rectangular to cylindrical coordinates, we can use the following three equations:

\[ A_\rho = \sqrt{A_x^2 + A_y^2} \] (Equation 2.15)

\[ \phi = \tan^{-1}\left(\frac{A_y}{A_x}\right) \] (Equation 2.16)

\[ A_z = A_z \] (Equation 2.17)

Equation 2.17 is only included for clarity, since no conversion is needed for the z-component when converting between rectangular and cylindrical coordinates.

To convert from cylindrical to rectangular coordinates, we use the following three equations:

\[ A_x = A_\rho \cos\phi \] (Equation 2.18)

\[ A_y = A_\rho \sin\phi \] (Equation 2.19)

\[ A_z = A_z \] (Equation 2.20)
Again, Equation 2.20 carries no useful information other than serving as a reminder that no conversion is needed.

Example 2.1: What is \( \mathbf{A}=3\mathbf{a}_x+2\mathbf{a}_y+1\mathbf{a}_z \) in cylindrical coordinates?

Example 2.2: What is \( \mathbf{B}=3\mathbf{a}_\rho+2\mathbf{a}_z \) (with \( \phi=35^\circ \)) in rectangular coordinates?

2.5 Converting Functions Between Rectangular and Cylindrical Systems

In addition to converting a vector between rectangular and cylindrical coordinates, we will sometimes need to convert an entire function from one to the other. This is actually quite straightforward—simply replace each occurrence of the old coordinate variable with a corresponding function from the new coordinate variable according to the following equations:

\[
\begin{align*}
    x &= \rho \cos \phi \\
    y &= \rho \sin \phi \\
    z &= z \\
    \rho &= \sqrt{x^2 + y^2} \\
    \phi &= \tan^{-1} \left( \frac{y}{x} \right) \\
    z &= z
\end{align*}
\]  

(Equation 2.21)  

(Equation 2.22)
Of course, no actual conversion of \( z \) is necessary when converting a function between rectangular and cylindrical coordinates.

**Example 2.3:** Please convert \( f(x,y,z) = x^2 + 3y^3z \) into cylindrical coordinates.

**Example 2.4:** Please convert \( f(\rho,\phi,z) = \rho^2 + z\rho\cos(\phi) \) into rectangular coordinates.

### 2.6 The Spherical Coordinate System

Recall that when we studied the cylindrical coordinate system, we first “aimed” using \( \phi \), then we moved away from the \( z \) axis a certain amount (\( \rho \)), and then we moved straight upward in the \( z \) direction to reach our destination. In spherical coordinates, we first aim in the \( x\)-\( y \) plane using \( \phi \) (the heading), then we adjust our vertical direction using \( \theta \) (the elevation), and then we move outward in the specified direction using \( r \) (the range). This sequence is shown in Figure 5.

![Figure 2.5. Representing a vector in spherical coordinates.](image)

The relationships among the unit vectors in spherical coordinates are shown below:

\[
\begin{align*}
    a_r \times a_\theta &= a_\phi \\
    a_\theta \times a_\phi &= a_r \\
    a_\phi \times a_r &= a_\theta
\end{align*}
\] (Equation 2.23)
The differential volume, surface areas, and length are a bit more complicated to derive in spherical coordinates than they were in rectangular or cylindrical coordinates. Two of the sides (in the \( \theta \) and \( \phi \) directions) now require multiplication by \( r \), since each of them represents an angle in radians. Furthermore, an interaction between the \( \theta \) and \( \phi \) directions requires the inclusion of a \( \sin(\theta) \) factor in some of the elements. Here are the differential elements in spherical coordinates:

\[
dv = r^2 \sin \theta \cdot dr \cdot d\theta \cdot d\phi \quad \text{(Equation 2.24)}
\]

\[
ds_r = \pm r^2 \sin \theta \cdot d\theta \cdot d\phi \cdot \mathbf{a}_r \quad \text{(Equation 2.25)}
\]

\[
ds_{\theta} = \pm r \sin \theta \cdot dr \cdot d\phi \cdot \mathbf{a}_{\theta} \quad \text{(Equation 2.26)}
\]

\[
ds_{\phi} = \pm r \cdot dr \cdot d\theta \cdot \mathbf{a}_{\phi} \quad \text{(Equation 2.27)}
\]

\[
dl = dr \cdot \mathbf{a}_r + r \cdot d\theta \cdot \mathbf{a}_{\theta} + r \sin \theta \cdot d\phi \cdot \mathbf{a}_{\phi} \quad \text{(Equation 2.28)}
\]

2.7 Converting Vectors Between Rectangular and Spherical Systems

Again, since any point in three-dimensional space can be represented by either rectangular or spherical coordinates, we should be able to convert between these two representations. Here are the representations of a given vector in rectangular and spherical coordinates, respectively:

\[
\mathbf{A} = A_x \mathbf{a}_x + A_y \mathbf{a}_y + A_z \mathbf{a}_z \quad \text{(Equation 2.29)}
\]

\[
\mathbf{A} = A_r \mathbf{a}_r \ \text{with} \ \phi \ \text{and} \ \theta \quad \text{(Equation 2.30)}
\]

To convert from rectangular to spherical coordinates, we can use the following three equations:

\[
A_r = \sqrt{A_x^2 + A_y^2 + A_z^2} \quad \text{(Equation 2.31)}
\]

\[
\theta = \tan^{-1} \left( \frac{\sqrt{A_x^2 + A_y^2}}{A_z} \right) \quad \text{(Equation 2.32)}
\]
\[ \phi = \tan^{-1}\left( \frac{A_y}{A_x} \right) \]  

(Equation 2.33)

To convert from spherical to rectangular coordinates, we use the following three equations:

\[ A_x = A_r \sin \theta \cos \phi \]  

(Equation 2.34)

\[ A_y = A_r \sin \theta \sin \phi \]  

(Equation 2.35)

\[ A_z = A_r \cos \theta \]  

(Equation 2.36)

**Important Note:** There are equations that can be used to convert between cylindrical and spherical coordinates, but this is a pretty unusual task to perform. If you are required to do so, it is probably best to just convert to rectangular coordinates as an intermediate step, then convert to the desired coordinates from rectangular.

**Example 2.5:** What is \( \mathbf{A} = 3\mathbf{a}_x + 2\mathbf{a}_y + 1\mathbf{a}_z \) in spherical coordinates?

**Example 2.6:** What is \( \mathbf{B} = 5\mathbf{a}_\rho \) (with \( \theta = 35^\circ \) and \( \phi = 60^\circ \)) in rectangular coordinates?

### 2.8 Converting Functions Between Rectangular and Spherical Systems

Just as we saw with cylindrical coordinates, we can also convert functions in either rectangular or spherical coordinates into the other coordinate system using the following sets of equations. Everywhere the old variable appears in the original equation, it is replaced by these functions:
\begin{align*}
x &= r \sin \theta \cos \phi \\
y &= r \sin \theta \sin \phi \\
z &= r \cos \theta \quad \text{(Equation 2.37)} \\
\end{align*}

\begin{align*}
r &= \sqrt{x^2 + y^2 + z^2} \\
\theta &= \tan^{-1} \left( \frac{\sqrt{x^2 + y^2}}{z} \right) \\
\phi &= \tan^{-1} \left( \frac{y}{x} \right) \quad \text{(Equation 2.38)}
\end{align*}

\textbf{Example 7:} Please convert \( f(x,y,z) = x^2 + 3y^3z \) into spherical coordinates.

\textbf{Example 8:} Please convert \( f(r,\theta,\phi) = r^2 + r \cos(\theta) \) into rectangular coordinates.

2.9 \textbf{Summary}

- Points in three-dimensional space can be represented by a linear combination of three orthogonal unit vectors.
- Rectangular, cylindrical, and spherical coordinate systems are the most common ways to select those unit vectors.
- In each of these coordinate systems, we can consider a differential element (an infinitesimally small cube) to calculate the differential volume, differential surface areas, and differential length. Each of these quantities will be useful when solving problems in electromagnetic fields.
- We will always use a right-handed coordinate system to solve our problems.
- We can convert either a vector or a function between rectangular and cylindrical coordinates.
- We can also convert either a vector or a function between rectangular and spherical coordinates.
- To convert between cylindrical and spherical coordinates, it is easiest to convert to rectangular coordinates as an intermediate step.