Chapter 16: Faraday’s Law of Induction

**Chapter Learning Objectives:** After completing this chapter the student will be able to:

- Use Faraday’s Law to determine the voltage appearing around a loop that is subject to a time-varying magnetic flux.
- Use Faraday’s Law of Induction to determine the current induced in a coil that is subject to a time-varying magnetic flux.
- Calculate the voltage induced across a rotating loop of wire.

You can watch the video associated with this chapter at the following link:

**Historical Perspective:** Michael Faraday (1791-1867) was a British scientist who made tremendous contributions to electromagnetic field theory. His work in electromagnetic induction later became the foundation of the work that ultimately led to the establishment of Maxwell’s Equations.

We have seen that voltages create currents, and currents induce magnetic fields. Today, we will see that changing magnetic fields also induce (create) voltages. **Faraday’s law** shows us that the voltage induced around a closed conductor is equal to the time derivative of the magnetic flux through that conductor, as shown in Equation 16.1. (We will explain the negative sign in just a little bit.)

\[
V(t) = -\frac{d\Psi_m(t)}{dt}
\]

(Equation 16.1)

A typical experimental setup for this measurement is shown in Figure 16.1.

This induced voltage can be measured by a voltmeter, but it is more common to include a resistor in the loop and then to measure the voltage across that resistor. Otherwise, there is no place for the induced voltage to dissipate.

There are four basic ways to change the magnetic flux through a loop of wire:

1. Change the magnitude of the magnetic field.
2. Change the direction of the magnetic field.
3. Change the shape or size of the loop.
4. Change the orientation of the loop.

We will see examples of each of these options in this chapter.

One question is which direction does the voltage (and therefore the current) appear? The answer to this comes from **Lenz’s Law**, which states that “the induced voltage will create a current that opposes the direction of the change in the magnetic flux.” The negative sign in Faraday’s Law comes from Lenz’s Law, since when it is used correctly with vector calculus, the negative sign will take care of the direction for you. I would suggest that you use the right-hand grip rule to help determine the direction of the voltage and current. Given this suggestion, I will omit the negative sign from Faraday’s Law and will simply use Lenz’s Law to determine the direction.
Example 16.1: Consider the following setup, in which the magnetic field increases in strength from 3T to 6T over the course of two seconds. How much voltage is induced across the resistor, and in which direction does the induced current flow?

![Diagram](image)

In previous chapters, we have discussed the relationship between $\Psi_m$ and $B$ somewhat informally, saying that $\Psi_m = B \cdot A$. However, as things get more complicated, we will need to be more mathematically precise. Equation 16.2 shows that $\Psi_m$ can be calculated as the surface integral of $B$ with respect to $dS$.

$$\Psi_m(t) = \int_{\Delta S} B \cdot dS$$

(Equation 16.2)

Also, you should be aware that if there are multiple loops in a coil of wire, then Faraday’s Law will have an additional $N$ factor to account for the sum of the voltage across each of those $N$ loops:

$$V(t) = N \frac{d\Psi_m(t)}{dt}$$

(Equation 16.3)

Remember to use Lenz’s Law to determine the direction of current flow, since the negative sign has been omitted from this equation.

Example 16.2: The direction of the magnetic field changes from perpendicular to the coil to 30° away from perpendicular to the coil over the course of three seconds. There are 10 turns on the coil. How much voltage is induced across the resistor, and in which direction does the induced current flow?

![Diagram](image)
**Example 16.3:** The moving bar in the following figure completes the fourth side of a loop. The surface area of the loop increases as the bar moves toward the right. What is the induced voltage in the resistor? Please account for the polarity of the voltage shown when indicating the sign of your answer.

![Diagram of moving bar completing a loop](image)

**Example 16.4:** Consider the following square loop, which has a time-varying magnetic flux density passing through it. There are 100 turns on the coil, and the magnetic field is 45° away from perpendicular to the surface of the loop. What are the magnitude and direction of the induced voltage and current?

\[ B = 10 \cdot \sin(2\pi \cdot 60t) \, \text{T} \]

![Diagram of square loop with magnetic field](image)

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### 16.2 Faraday’s Law Applied to Inductors

You may recall from the previous chapter that I promised to prove the V-I relationship for inductors. It’s time to fulfill that promise. Recall that an inductor is simply a coil of wire around a ferromagnetic core, as shown in Figure 16.2.
Figure 16.2. An inductor

In chapter 15, we proved that the total magnetic flux and inductance of such a coil can be written as follows:

\[ \Psi_m = \frac{\mu NI}{d} \pi a^2 \]  
(Copy of Equation 15.6)

\[ L = \frac{\mu N^2}{d} \pi a^2 \]  
(Copy of Equation 15.8)

We will now substitute Equation 15.6 into Faraday's Law, Equation 16.3:

\[ V(t) = N \frac{d\Psi_m(t)}{dt} = N \frac{d}{dt} \left[ \frac{\mu NI}{d} \pi a^2 \right] \]  
(Equation 16.4)

Everything inside the brackets is constant except I, so we can simplify this as:

\[ V(t) = \left[ \frac{\mu N^2}{d} \pi a^2 \right] \frac{dI}{dt} \]  
(Equation 16.5)

Now, the quantity in the brackets matches the right side of Equation 15.8, so we can replace it all with L:

\[ V(t) = L \frac{dI}{dt} \]  
(Equation 16.6)

Thus, the V-I relationship for inductors is really just a particular application of Faraday's Law. The voltage appearing across the terminals of the inductors is an example of induced voltage. This voltage is being induced by the magnetic flux density passing through the coil, which is in turn being created by the current flowing through the wire.
Faraday’s Law Applied to Transformers

Last chapter, we also studied transformers (also called coupled or mutual inductors). At the time, I explained to you how transformers are used to “step up” or “step down” voltage in order to facilitate transmission and distribution of electrical power from power plants to end-users. Now we can study transformers in more detail using Faraday’s Law.

Consider the transformer shown in Figure 16.4. There are $N_1$ turns on the left coil and $N_2$ turns on the right coil. We will now derive relationships between $V_1$ and $V_2$ and between $I_1$ and $I_2$.

This derivation will be surprisingly simple. We know, of course, that the same magnetic flux $\Psi_m$ flows through both of the coils. We can therefore write two Faraday’s Law equations, one for each coil:

$$V_1 = N_1 \frac{d\Psi_m}{dt} \quad (\text{Equation 16.7})$$
$$V_2 = N_2 \frac{d\Psi_m}{dt} \quad (\text{Equation 16.8})$$

If we now solve each of these two equations for the derivative of magnetic flux and set them equal to each other, we find that:

$$\frac{V_2}{V_1} = \frac{N_2}{N_1} \quad (\text{Equation 16.9})$$

Notice, then, that the ratio of the two transformer voltages depends only on the ratio of the number of turns on the two coils. This quantity is typically referred to as the “turns ratio.”
If we also assume that there is no power being lost inside the transformer, then the input power (on the left) must also be equal to the output power (on the right). [This assumption is valid to within about 3-5% for a modern transformer design.] Since power is current multiplied by voltage, we obtain:

$$I_1 \cdot V_1 = I_2 \cdot V_2$$

(Equation 16.10)

Solving this equation for $I_1/I_2$, we find that this is equal to $V_2/V_1$, which is the turns ratio:

$$\frac{I_1}{I_2} = \frac{V_2}{V_1} = \frac{N_2}{N_1}$$

(Equation 16.11)

So, if the turns ratio is greater than one (which is true for a “step-up” transformer), then $V_2 > V_1$, but the tradeoff is that $I_2 < I_1$.

Finally, we can consider the resistance seen on the primary (left) coil and on the secondary (right) coil:

$$R_{primary} = \frac{V_1}{I_1}$$

(Equation 16.12)

$$R_{secondary} = \frac{V_2}{I_2}$$

(Equation 16.13)

We can next divide Equation 16.12 by Equation 16.13, simplify the result, and substitute Equations 16.9 and 16.11 into the result:

$$\frac{R_{primary}}{R_{secondary}} = \frac{V_1}{V_2} \cdot \frac{I_2}{I_1} = \left( \frac{N_1}{N_2} \right)^2$$

(Equation 16.14)

This shows that a transformer will effectively modify the resistance “seen” on the primary coil when a resistor is attached to the secondary coil. This can be used for “impedance matching” when a load has a resistance that is too small and would otherwise draw too much current from a source. Audio applications, in particular, use impedance matching transformers quite frequently.

Example 16.5: A speaker has an impedance of 6Ω. A 12V AC source must not deliver more than 100mA of current to its load. Design an impedance-matching transformer to make this possible.
Faraday’s Law Applied to Electrical Generators

Electrical generators are literally the driving force behind the entire field of electrical and computer engineering. With the exception of solar power, essentially all electricity we use is derived from spinning a coil of wire in a strong magnetic field. Hydroelectric dams, wind turbines, coal power plants, and gas-powered generators all use their incoming power to spin a rotor inside a magnetic field, and the output power comes from the induced voltage across the spinning rotor.

Figure 16.4 shows a schematic representation of an electrical generator. A magnet (either permanent or, more likely, an electromagnet) creates a magnetic flux density \( B \), and a rotor is forced to spin within the region of that flux density.

![Figure 16.4. An Electrical Generator](image)

The red and blue sides of the loop have no physical significance—they are only used to help you understand how the rotor is rotating.

This is clearly an example of a constant magnetic field and a constant size and shape of the loop, but the angle of the loop is changing. We can now determine the voltage appearing across the terminals of the loop. We will begin by calculating the magnetic flux, which requires a dot product between the constant magnetic flux density and the rotating coil. The dot product will introduce a cosine term:

\[
\Psi_m(t) = \int_A \mathbf{B} \cdot d\mathbf{S} = B \cdot A \cdot \cos(\theta) \quad \text{(Equation 16.15)}
\]

We can now apply Faraday’s Law to calculate the voltage across the output terminals of the generator, substituting Equation 16.15 for the magnetic flux:
\[ V(t) = N \frac{d\Psi_m(t)}{dt} = N \frac{d}{dt}(B \cdot A \cdot \cos(\theta)) \]  
(Equation 16.16)

B and A are constant, so they can be brought outside the derivative, and the derivative of \( \cos(\theta) \) is found using the chain rule:

\[ V(t) = NBA \cdot \frac{d\cos(\theta)}{dt} = NBA \cdot (-\sin(\theta)) \frac{d\theta}{dt} \]  
(Equation 16.17)

We know that the derivative of the angle is the angular velocity \( \omega \), so the final answer is:

\[ V(t) = -NBA\omega \sin(\theta) \]  
(Equation 16.18)

The negative sign is really irrelevant, since this is a sinusoidal quantity. We see that the magnitude of the generated voltage will increase with the number of turns on the coil, the strength of the magnetic flux, the area of the loop, and the speed with which the coil is turned. All of these are intuitive results.

This calculation also shows one of the main reasons why AC power is used so frequently—it is very easy to generate. Just spin a coil of wire in a magnetic field, and you get AC power coming out! By comparison, it is actually quite difficult to generate DC power. It is almost always easier to generate AC and then convert it to DC, even if the converter costs us a few percentage points of loss.

Example 16.6: What is the magnitude of the voltage output by a generator with a cross-sectional loop area of 2m², 1000 turns, \( B=10T \), and the rotor rotates at 3600RPM?

16.5 Final Form of Faraday's Law of Induction

The version of Faraday’s Law that we have been using in this chapter is complete, and there is nothing else that needs to be added to it. However, there is some benefit to restating it in a more mathematically complex manner. (Doing so will make it more evident how Faraday’s Law and the soon-to-be-completed Ampere’s Law are mathematical complements of each other. Let’s
spruce up Faraday’s Law before it is elevated to its rightful position among Maxwell’s Equations.

Recall from chapter 7 that we can express voltage as a line integral of the electric field:

\[ V(t) = \int \mathbf{E} \cdot d\mathbf{l} \]  \hspace{1cm} \text{(Equation 16.19)}

We can substitute this equation into the left side of Equation 16.1:

\[ \int \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Psi_m(t)}{dt} \]  \hspace{1cm} \text{(Equation 16.20)}

If we now substitute Equation 16.2 into the derivative on the right side, we obtain:

\[ \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_{\Delta_s} \mathbf{B} \cdot d\mathbf{s} \]  \hspace{1cm} \text{(Equation 16.21)}

This is the official version of the Integral Form of Faraday’s Law of Induction, which is hereby inducted as the third of Maxwell’s Equations.

Stoke’s Theorem (whose proof is 100% valid but also long and tedious) says that the closed line integral around a region of space is equal to the surface integral of the curl over the same region:

\[ \oint \mathbf{E} \cdot d\mathbf{l} = \int_{\Delta_s} \nabla \times \mathbf{E} \cdot d\mathbf{s} \]  \hspace{1cm} \text{(Equation 16.22)}

We can apply this equation to the left side of Equation 16.21, and we can bring the derivative inside the integral on the right side. (A mathematician would warn you that doing so requires certain assumptions to be true, and I can assure you that they are true in this circumstance.)

\[ \int_{\Delta_s} (\nabla \times \mathbf{E}) \cdot d\mathbf{s} = -\int_{\Delta_s} \frac{d\mathbf{B}}{dt} \cdot d\mathbf{s} \]  \hspace{1cm} \text{(Equation 16.23)}

Since both sides of this equation are a surface integral over the same surface, the only way it can be true is if the quantities being integrated on both sides are equal. Thus, we can write the Point Form of Faraday’s Law of Induction:

\[ \nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt} \]  \hspace{1cm} \text{(Equation 16.24)}
Summary

- Faraday’s Law states that changing magnetic flux will induce a voltage around a loop:

\[ V(t) = N \frac{d\psi_m(t)}{dt} \]

- Lenz’s Law (sometimes represented by a negative sign in Faraday’s Law) says that the induced voltage will be in a direction that opposes the changing magnetic flux.

- When the magnetic flux density is not perpendicular to the loop, you may need to use the formal definition of magnetic flux:

\[ \psi_m(t) = \int_{\Delta s} B \cdot dS \]

- Faraday’s Law can be applied in a variety of circumstances.

Inductors:

\[ V(t) = L \frac{dI}{dt} \]

Transformers:

\[ \frac{V_2}{V_1} = \frac{N_2}{N_1} \]

\[ \frac{I_1}{I_2} = \frac{N_2}{N_1} \]

\[ \frac{R_{pri}}{R_{sec}} = \left( \frac{N_1}{N_2} \right)^2 \]

Generators:

\[ V(t) = -NA\omega \sin(\theta) \]

- The formal, official version of Faraday’s Law can be written in integral and point form:

\[ \oint E \cdot dl = -\frac{d}{dt} \int_{\Delta s} B \cdot ds \]

\[ \nabla \times E = -\frac{dB}{dt} \]