Chapter 14: Magnetic Materials and Magnetic Circuits

Chapter Learning Objectives: After completing this chapter the student will be able to:

- Explain the difference between diamagnetic, paramagnetic, and ferromagnetic materials.
- Calculate the magnetic field intensity from the magnetic flux density and vice versa.
- Explain the hysteresis loop in the B-H curve.
- Analyze a magnetic circuit to determine the magnetic flux at each point in the circuit.

You can watch the video associated with this chapter at the following link:

Historical Perspective: Oliver Heaviside (1850-1925) was a self-taught electrical engineer, physicist, and mathematician who introduced the use of complex numbers to solve AC circuits. He also did a great deal of work in electromagnetic fields, and he coined the terms “admittance,” “conductance,” “impedance,” “inductance,” “permeability,” “permittivity,” and “reluctance.”

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There are three major types of materials that respond to magnetic fields: paramagnetic, diamagnetic, and ferromagnetic. The first two of these are relatively weak, while the third is quite strong and forms the foundation of much of electrical and computer engineering.

Diamagnetism is a quantum mechanical effect that causes electron orbitals to slightly shift in response to an externally applied magnetic field. This shift occurs in such a manner that the magnetic field is partially expelled from inside the material. This slightly increases the potential energy of the system, which means that the diamagnetic material is weakly repelled by a magnetic field. Diamagnetism is actually a characteristic of all materials, but it is often overwhelmed by stronger effects (paramagnetism and especially ferromagnetism). Common diamagnetic materials include water, bismuth, mercury, and silver. Interestingly, superconductors perfectly expel magnetic fields from their interior, and so they can be considered “perfectly diamagnetic” materials. A superconductor will therefore strongly repel a magnet, creating a form of magnetic levitation.

Paramagnetic materials are weakly attracted to externally applied magnetic fields due to the magnetic field associated with unpaired valence electrons. You may recall that each electron orbital can contain two electrons, one with “spin up” and the other with “spin down.” It is not precisely correct to that electrons spin, but they do each exhibit a relatively weak magnetic field. When the orbital contains two electrons, these magnetic fields cancel. But when the orbital contains only one electron, it cannot be cancelled. In this case, it responds to an external magnetic field by distorting the orbital’s shape, which then causes the atom to align with the magnetic field. Common paramagnetic materials including oxygen, aluminum, and titanium. Although paramagnetism is relatively weak, it is strong enough to overwhelm the diamagnetic force that is also present in such materials.

Ferromagnetic materials are very strongly attracted to externally applied magnetic fields. As was discussed in section 11.3, ferromagnetic materials (iron, nickel, cobalt, and their alloys) respond to the presence of an externally applied magnetic field by changing the size and direction of the magnetic domains that are present in these materials. As these domains shift in response to the externally applied magnetic field, the material develops an internal magnetic field (called the “magnetization”) and the material is then very strongly attracted to the externally applied magnetic field.

As an analogy to explain the relative magnitude of these three effects, if diamagnetism is represented by the weight of a feather, then paramagnetism would be the weight of the bird, and ferromagnetism is the weight of a jet airplane.
14.2 Magnetization

Where chapter 11 presented a qualitative discussion of the response of ferromagnetic materials to an externally applied magnetic field, we will now develop a quantitative description that will allow us to significantly expand our understanding of magnetic fields.

Recalling that each atom in a ferromagnetic material exhibits a magnetic dipole moment ($m$), we can define the magnetization of the material ($M$) to be the magnetic dipole moment of an infinitesimally small volume:

$$ M = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \sum_{j=1}^{j=n} m_j \quad \text{(Equation 14.1)} $$

Thus, we can also consider the magnetization $M$ to be the magnetic dipole per unit volume.

As you may recall from the discussion of dielectric materials in section 8.2, the polarization inside a dielectric material is proportional to the electric field:

$$ P = \varepsilon_0 \chi_e E \quad \text{(Copy of Equation 8.8)} $$

This polarization field contributes to the strength of the electric field inside the material, and we introduce a new quantity (electric flux density $D$) to include its effect:

$$ D = \varepsilon_0 \cdot (1 + \chi_e) E \quad \text{(Copy of Equation 8.9)} $$

$$ D = \varepsilon_0 \varepsilon_r E \quad \text{(Copy of Equation 8.11)} $$

We can perform a similar analysis for the effect of magnetization in ferromagnetic materials on an externally applied magnetic field. Just as the concept of $E$ was expanded to include $P$ and the combination of $E$ and $P$ was represented by $D$, we will see that the concept of $B$ will be expanded to include $M$, and the combination of $B$ and $M$ will be represented by a new quantity, $H$. 
14.3 Magnetic Field Intensity and Permeability

We will now define a new quantity $H$, known as the magnetic field intensity, and which has units of A/m. $H$ will include the effect of externally applied magnetic flux density $B$ as well as the magnetization inside the material, $M$:

$$ H = \frac{B}{\mu_0} - M \quad \text{(Equation 14.2)} $$

We can now re-write Ampere’s Law (just as we re-wrote Gauss’s Law) to use $H$ instead of $B$:

$$ \int \mathbf{H} \cdot d\mathbf{l} = I_{enc} \quad \text{(Equation 14.3)} $$

This equation is typically referred to as Ampere’s Circuital Law in order to distinguish it from Ampere’s Force Law, which we saw in section 13.4. Notice that the $\mu_0$ term is included in the left side of the equation now, rather than on the right as it was originally shown in Equation 11.9.

This expansion of Ampere’s Circuital Law brings it one step closer to its final form (which will be one of Maxwell’s Equations), but there is still one more addition to be made. (Spoiler alert: The final form of this equation will first appear in Chapter 17).

You may recall that the strength of the polarization field inside a dielectric material is proportional to the externally applied electric field. We can make the same assumption here, that the magnetization is proportional to the magnetic field intensity. (As we will shortly see, this assumption is imperfect, but it does work well for relatively weak magnetic fields.)

$$ M = \chi_m \cdot H \quad \text{(Equation 14.4)} $$

The constant $\chi_m$ is referred to as the magnetic susceptibility, and it is a physical characteristic of the material being considered. It represents how susceptible the material is to creating a magnetization field when an external magnetic field is applied to it.

We can now re-arrange Equation 14.2 to obtain:

$$ B = \mu_0 \cdot (H + M) \quad \text{(Equation 14.5)} $$

Substituting Equation 14.4 into Equation 14.5, we find:
\[ \mathbf{B} = \mu_o \cdot (\mathbf{H} + \chi_m \cdot \mathbf{H}) \]  
(Equation 14.6)

Factoring \( \mathbf{H} \) out of both terms and using a new quantity, the relative permeability \( (\mu_r) \), we obtain:

\[ \mathbf{B} = \mu_o \cdot (1 + \chi_m) \cdot \mathbf{H} = \mu_o \cdot \mu_r \cdot \mathbf{H} = \mu \cdot \mathbf{H} \]  
(Equation 14.7)

Notice that the total permeability, \( \mu \), is the product of the relative permeability \( \mu_r \) and the permeability of free space, \( \mu_0 \). The relative permeability is defined to be:

\[ \mu_r = 1 + \chi_m \]  
(Equation 14.8)

Table 14.1 shows the relative permeability for a variety of different ferromagnetic materials. All non-ferromagnetic materials will have a relative permeability very close to 1.

<table>
<thead>
<tr>
<th>Material</th>
<th>Relative Permeability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Supermalloy</td>
<td>1,000,000</td>
</tr>
<tr>
<td>Iron (99.95% pure)</td>
<td>200,000</td>
</tr>
<tr>
<td>Cobalt-Iron Alloy</td>
<td>18,000</td>
</tr>
<tr>
<td>Permalloy</td>
<td>8000</td>
</tr>
<tr>
<td>Iron (99.8% pure)</td>
<td>5000</td>
</tr>
<tr>
<td>Cobalt</td>
<td>600</td>
</tr>
<tr>
<td>Nickel</td>
<td>300</td>
</tr>
<tr>
<td>Carbon Steel</td>
<td>100</td>
</tr>
</tbody>
</table>

FAQ Example 14.1: In Example 11.6, we found that the magnetic flux density \( B \) inside an air-filled toroidal solenoid with \( N \) turns of wire, radius \( a \), circumference \( d \), and current \( I \) was \( B = \frac{\mu_0 N I}{d} \). How much does the flux density increase if the core is filled with cobalt rather than air?
Example 14.2: If $H=100\,\text{A/m}$ in a solid piece of Permalloy, what are the magnetization field and the magnetic flux density?

### 14.4 Ferromagnetic Hysteresis and the Memory Effect

As shown in Equation 14.4, we often assume that the magnetization $M$ is proportional to the magnetic field intensity $H$. After some algebra, this leads to the conclusion in Equation 14.7 that the magnetic flux density is also proportional to the magnetic field intensity. This is quite true for low-intensity magnetic fields. However, we often use high-intensity magnetic fields, and that can invalidate this assumption for ferromagnetic materials.

Figure 14.1 shows the true relationship between $B$ and $H$ for ferromagnetic materials. Notice the arrows on this figure—these are present because the past history of the applied magnetic field has an impact on the current state of the magnetization field. This “memory effect” of ferromagnetic materials is known as **hysteresis**. As described in Section 11.3, this occurs because externally applied magnetic fields cause changes to the domains that persist even after the magnetic field has been removed.

![Figure 14.1. The B-H curve illustrates hysteresis](image)

Let’s take a trip around the hysteresis loop in Figure 14.1. We will begin at point A, which is where the material has either never been subject to a magnetic field or it has been cancelled or has faded. As a positive field is applied, we move along the dotted line from A to B. Eventually, the magnetic flux density will saturate, which means that the curve flattens out once most or all
of the domains are pointing in the “correct” direction. Moving from B to C, we remove the external field (H), but the magnetic flux density B still remains non-zero. Thus, it remembers that it was previously subjected to a magnetic field. At point D, we have applied a negative field sufficiently strong to return the material to its unmagnetized state. If we then continue to increase the negative field, it will again saturate a point E. At point F, the external field has been removed, but the material still has a non-zero magnetic flux density, while at point G, a sufficiently positive magnetic field has been applied to cancel out the memory effect. If the positive field continues to increase, we return to point B and close the loop.

Although hysteresis can be a hassle to deal with, it is also a huge benefit in some circumstances. Magnetic data storage, for example, relies on the memory effect to store binary 0s and 1s. Material scientists have worked very hard to optimize the hysteresis loop for data memory storage, and the result is shown in Figure 14.2. Notice that passing through point A, we can then remove the magnetic field, and the system will remain magnetized a point B (a binary “1”). The magnetic flux density all along the top of the curve is nearly constant, and it will remain that way until a sufficiently strong negative field is applied, which will pass through point C, then to D and then E (a binary “0”). Such devices are very stable and capable of storing data for years or even decades.

![Figure 14.2. A hysteresis loop optimized for data memory storage.](image)

**14.5 Magnetic Circuits**

The energy stored inside a magnetic field can be represented as follows:

$$ W = \frac{1}{2} \int (H \cdot B) dV $$

(Equation 14.9)
If we once again assume that the field strength is relatively low, this Equation can be simplified as follows:

\[ W = \frac{1}{2} \frac{B^2}{\mu_0 \mu_r} V \quad \text{(Equation 14.10)} \]

Notice that the \( \mu_r \) in the denominator of this equation indicates that the energy will be lower for a magnetic field to pass through a ferromagnetic material than to pass through a vacuum (or air). Physics dictates that systems will always evolve toward lower-energy states if possible, so we observe that (1) if possible, ferromagnetic materials will move toward regions of high magnetic fields, and (2) if possible, magnetic fields will tend to pass through ferromagnetic materials rather than through the air.

This tendency of magnetic fields to move through ferromagnetic materials is very similar to the “tendency” of electrical current to pass through a conductor rather than through the air. This tendency is so strong that we almost always blindly assume that the current will simply pass through the conductor rather than jumping through the air. We can make precisely the same assumption with magnetic fields.

The tendency of magnetic fields to remain inside ferromagnetic materials leads to what are called “magnetic circuits,” in which analysis of magnetic fields can be performed quite easily, often using tools similar to those for electric circuits. Consider Figure 14.3, which shows the simplest possible magnetic circuit: a coil of wire inducing magnetic fields to pass through a closed path of solid (ferromagnetic) material.

![Figure 14.3. A simple magnetic circuit.](image)
We can calculate the magnetic flux beginning with the new version of Ampere’s Circuital Law (Equation 14.3). Following the dotted line path in Figure 14.3, we obtain:

\[ H \cdot L = NI \]  
(Equation 14.11)

This can be solved for \( H \):  

\[ H = \frac{NI}{L} \]  
(Equation 14.12)

We can then use Equation 14.7 to find \( B \):  

\[ B = \mu \cdot H = \frac{\mu_0 \mu_r NI}{L} \]  
(Equation 14.13)

We will now introduce a new quantity, called “magnetic flux,” which is represented by the Greek letter \( \psi_m \). Magnetic flux is the magnetic flux density \( B \) multiplied by the surface area, just as current \( I \) is the current density \( J \) multiplied by the surface area.

\[ B = \frac{\Psi_m}{A} \]  
(Equation 14.14)

\[ \Psi_m = B \cdot A \]  
(Equation 14.15)

Combining Equation 14.13 with Equation 14.14, we obtain:

\[ \Psi_m = \frac{\mu_0 \mu_r NI A}{L} \]  
(Equation 14.16)

If we define the “reluctance” of the iron core to be:

\[ R = \frac{L}{A \mu_0 \mu_r} \]  
(Equation 14.17)

We can then rewrite Equation 14.16 as follows:

\[ \Psi_m = \frac{NI}{R} \]  
(Equation 14.18)
This can, in turn, be rewritten as:

\[ NI = \Psi_m \cdot R \]  

(Equation 14.19)

Consider this equation for a moment. The term NI, typically referred to as the “magnetomotive force,” is the driving force in the analysis. Without it, no action occurs. The reluctance \( R \) represents the material’s resistance to having magnetic fields, and the magnetic flux \( \Psi_m \) represents that quantity that flows as a result of NI in spite of \( R \). This is completely analogous to Ohm’s Law! In fact, we will find that we can convert any magnetic circuit into an analogous electric circuit by first calculating the reluctance of each section of the circuit (Equation 14.17), calculate the magnetomotive force NI, and then calculate the magnetic flux from Equation 14.19. Once the magnetic flux has been calculated, the magnetic flux density can be calculated from Equation 14.14.

**Example 14.3:** What is the magnetic flux density \( B \) for the magnetic circuit in Figure 14.3 if the cross-section is 10cm in each direction, the total path length around the loop is 50cm, there are 100 turns of wire, and each is carrying 1A?

We can also extend the analogy of magnetic circuits to electric circuits to include essentially all of the electric circuit analysis tools in our toolbox, including series and parallel combinations, Kirchhoff’s Laws, and mesh and nodal analysis. Simply calculate the reluctances, calculate the magnetomotive force(s), and then calculate the magnetic flux (and flux density, if necessary).

**Example 14.4:** Symbolically calculate the flux density present in the air gap of the following magnetic circuit. You should express your answer in a format similar to Equation 14.18.
Example 14.5: Repeat Example 14.3 if a 2mm air gap is present somewhere along the 50 cm loop. By what percent does the magnetic flux density decrease?

Example 14.6: Convert the following magnetic circuit into an electrical circuit analog. Write two mesh current equations for the magnetic flux in each mesh.

Summary

- Diamagnetic materials are weakly repelled by magnetic fields, paramagnetic fields are weakly attracted to magnetic fields, and ferromagnetic materials are very strongly attracted to magnetic fields.
- The magnetic field intensity $H$ (units A/m) is defined to include both the effect of an externally applied magnetic field and the magnetization field inside a solid material:

$$ H = \frac{B}{\mu_0} - M $$
A revised version of Ampere’s Circuital Law uses $H$ rather than $B$ when ferromagnetic materials are involved:

$$\oint H \cdot dl = I_{enc}$$

The magnetization field is proportional to the magnetic field intensity for relatively low intensities:

$$\mathbf{M} = \chi_m \cdot \mathbf{H}$$

This results in the magnetic flux density $B$ being proportional to the magnetic field intensity $H$:

$$\mathbf{B} = \mu_0 \cdot \mu_r \cdot \mathbf{H}$$

Ferromagnetic materials exhibit hysteresis, which means that they “remember” magnetic fields that have been applied to them in the past. This can be used to our advantage in magnetic memory devices.

Whenever possible, magnetic fields will remain in ferromagnetic materials rather than passing through the air. This allows us to study “magnetic circuits,” which can be analyzed using the same tools as electric circuits. Magnetomotive force, which takes the place of voltage, can be calculated by the product $NI$, and the reluctance of each branch, which takes the place of resistance, can be calculated by:

$$\mathcal{R} = \frac{L}{A\mu_0\mu_r}$$

Ohm’s Law for magnetic circuits can then be written as:

$$NI = \Psi_m \cdot \mathcal{R}$$

Where $\Psi_m$ is the total magnetic flux. Given this quantity, we can calculate the magnetic flux density according to:

$$B = \frac{\Psi_m}{A}$$