Chapter Learning Objectives: After completing this chapter the student will be able to:
- Explain the difference between scalars and vectors and when each should be used.
- Use unit vectors to represent two- and three-dimensional vector quantities in the rectangular coordinate system.
- Calculate the magnitude of a vector.
- Perform vector addition and subtraction.
- Calculate the dot product between two vectors using two different methods.
- Determine the angle between two vectors using the dot product.
- Calculate the cross product between two vectors using two different methods.

You can watch the video associated with this chapter at the following link:

Historical Perspective: Vector arithmetic first appeared in its current form in the late 19th century when Josiah Willard Gibbs and Oliver Heaviside independently developed vector analysis. They did so in an effort to apply Maxwell’s equations, which had been put together by James Clerk Maxwell in 1865. Maxwell’s equations describe the behavior of electromagnetic fields, so it is fair to say that vectors were invented for this subject.
### 1.1 Scalars vs. Vectors

Most of the quantities that we work with are scalars, which is simply a number without a direction. Temperature, volume, and electrical charge are examples of directionless (scalar) quantities.

However, most of the quantities in this class will have both a number (the magnitude) and a direction associated with them, so they are best represented using vectors. They are also typically three-dimensional quantities, which means that most of the vectors we will use will require three numbers (components) to represent them. Examples of vectors we will include current density, electric field intensity, and magnetic field intensity.

### 1.2 Unit Vectors and the Rectangular Coordinate System

Unit vectors have a magnitude or length of 1, and they point in an important direction. For example, a vector that is one meter long and points due north would be a good choice for a unit vector.

Vector quantities are typically represented as a weighted sum of three unit vectors. For example, we could define the location of an airplane relative to the landing strip by saying how far east/west it is, how far north/south it is, and how far above (or below!) the surface of the landing strip it is. Since we live in a three-dimensional universe, we will typically need three unit vectors to specify any location relative to another location.

We will represent unit vectors using a lowercase $a$ with a subscript indicating the direction of the unit vector. In printed copy, unit vectors will be listed with a bold font ($\mathbf{a}$), but when handwritten, they are given a hat over the vector ($\hat{a}$).

In rectangular coordinates, the three unit vectors are $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$, each of which has a magnitude of one and points in the direction indicated by its subscript. Any other vector in three-dimensional space can be written as a weighted sum of these three unit vectors, as shown in Figure 1.1.
Example 1.1: Using rectangular coordinates, what is the vector that points from (-1, -2, 3) to (2, 4, 1)?

1.3 Vector Magnitude

The magnitude of a vector can be found using a form of the Pythagorean Theorem, as shown in Equation 1.1:

$$|\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

(Equation 1.1)

Here, $A_x$, $A_y$, and $A_z$ are the three quantities associated with the $x$, $y$, and $z$ coordinates, respectively. The quantity $|\mathbf{A}|$ represents the overall magnitude or size of the vector $\mathbf{A}$.

Sometimes, you will need to find a unit vector in a particular direction. Remembering that a unit vector is just a normal vector with a magnitude of one, we can divide the vector by its own magnitude in order to convert it to a unit vector, as shown in Equation 1.2:

$$\mathbf{a}_A = \frac{\mathbf{A}}{|\mathbf{A}|}$$

(Equation 1.2)
Example 1.2: Find a unit vector in the direction of the solution to Example 1.1.

1.4 Vector Addition and Subtraction

Vector addition and subtraction are quite simple in rectangular coordinates: You just add or subtract the three components of the vectors being added or subtracted, as shown in Equations 1.3 and 1.4.

\[
\begin{align*}
A + B &= (A_x + B_x)\mathbf{a}_x + (A_y + B_y)\mathbf{a}_y + (A_z + B_z)\mathbf{a}_z \quad (\text{Equation 1.3}) \\
A - B &= (A_x - B_x)\mathbf{a}_x + (A_y - B_y)\mathbf{a}_y + (A_z - B_z)\mathbf{a}_z \quad (\text{Equation 1.4})
\end{align*}
\]

It is also possible to visualize vector addition using the parallelogram law, which says to put the two vectors to be added head to tail, then the sum will be a vector from the free head to the free tail. This is illustrated in Figure 1.2.

![Figure 1.2. The parallelogram law for vector addition.](image)

Example 1.3: If \(A = 3a_x + 2a_y + 1a_z\) and \(B = 4a_x + 1a_y + 5a_z\), find \(A + B\) and \(A - B\).
Multiplying vectors is a bit more complicated than adding and subtracting them. There are actually two ways to multiply vectors, the dot product and the cross product. The dot product of two vectors will yield a scalar number, while the cross product will yield another vector. Both of these quantities can be useful in different circumstances, and it is important to know how to calculate both.

The fundamental definition of a dot product is:

\[
\mathbf{A} \cdot \mathbf{B} = |A| \cdot |B| \cdot \cos \theta = AB \cdot \cos \theta \quad \text{(Equation 1.5)}
\]

Here, unbolded letters are used as a shorthand notation for the magnitude of a vector in order to simplify the appearance of the equation.

The \( \cos \theta \) term represents the degree to which the two angles are pointing in the same direction. If \( \theta = 0 \), then the angles line up perfectly, and the dot product is just the product of the two magnitudes. This is illustrated in Figure 1.3:

![Figure 1.3](image)

**Figure 1.3.** A dot product is the magnitude of a vector multiplied by the projection of another onto it.

There is also another way to calculate the dot product if the vectors are both in rectangular form:

\[
\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z \quad \text{(Equation 1.6)}
\]

**Example 1.4:** If \( \mathbf{A} = 2\mathbf{a}_x \) and \( \mathbf{B} = 1\mathbf{a}_x + 2\mathbf{a}_y \), find \( \mathbf{A} \cdot \mathbf{B} \) using equation 1.5.
Example 1.5: If \( \mathbf{A} = 2\mathbf{a}_x \) and \( \mathbf{B} = 1\mathbf{a}_x + 2\mathbf{a}_y \), find \( \mathbf{A} \cdot \mathbf{B} \) using equation 1.6.

### 1.6 Angle Between Vectors

If you are given the two vectors in rectangular coordinates, you can immediately calculate the magnitude of each using Equation 1.2, then calculate the dot product using equation 1.6. These quantities can then be inserted into equation 1.6, and the only unknown quantity will be \( \cos \theta \). This allows you to quickly determine the angle between two vectors in rectangular form by combining the two methods for calculating the dot product:

\[
\theta = \cos^{-1} \left( \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}| \cdot |\mathbf{B}|} \right) = \cos^{-1} \left( \frac{A_x B_x + A_y B_y + A_z B_z}{\sqrt{A_x^2 + A_y^2 + A_z^2} \cdot \sqrt{B_x^2 + B_y^2 + B_z^2}} \right) \quad \text{(Equation 1.7)}
\]

Example 1.6: What is the angle between the vectors \( \mathbf{A} = 3\mathbf{a}_x + 2\mathbf{a}_y + 1\mathbf{a}_z \) and \( \mathbf{B} = 4\mathbf{a}_x + 1\mathbf{a}_y + 5\mathbf{a}_z \) ?

### 1.7 Cross Products

The second type of vector multiplication is the cross product, which results in another vector. The resulting vector has a magnitude that can be found from Equation 1.8:

\[
|\mathbf{A} \times \mathbf{B}| = AB \cdot \sin \theta \quad \text{(Equation 1.8)}
\]
The direction of the cross product vector can be found according to the right-hand rule:

![Right-hand rule](https://en.wikipedia.org/wiki/Right-hand_rule)

**Figure 1.4.** The right-hand rule is used to determine the direction of the cross product.  
(Copyright information: CC BY-SA 3.0: [https://en.wikipedia.org/wiki/Right-hand_rule](https://en.wikipedia.org/wiki/Right-hand_rule))

As with the dot product, there is also a shortcut for finding the cross product if the original vectors are in rectangular form:

\[ \mathbf{A} \times \mathbf{B} = \begin{vmatrix} a_x & a_y & a_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \]  
(Equation 1.9)

This very compact and memorable form of the vector (unit vectors on the top row, components of \( \mathbf{A} \) on the second row, and components of \( \mathbf{B} \) on the third row) can be expanded to a more easily calculable result using a trick for the 3x3 determinant. Repeat the first two columns, then form the six diagonal products shown in Figure 1.5. Those pointing downward become positive terms, while those pointing upward become negative terms.

![3x3 determinant trick](https://example.com/det_trick)

**Figure 1.5.** A shortcut for finding a 3x3 determinant.

Thus, we find that:

\[ \mathbf{A} \times \mathbf{B} = A_y B_z \mathbf{a}_x + A_z B_y \mathbf{a}_y + A_x B_z \mathbf{a}_z - A_z B_x \mathbf{a}_x - A_y B_z \mathbf{a}_y - A_x B_y \mathbf{a}_z \]  
(Equation 1.9)

Or, grouping components from the same direction, we get:

\[ \mathbf{A} \times \mathbf{B} = \left( A_y B_z - A_z B_y \right) \mathbf{a}_x + \left( A_z B_x - A_x B_z \right) \mathbf{a}_y + \left( A_x B_y - A_y B_x \right) \mathbf{a}_z \]  
(Equation 1.10)
Notice that, although the dot product yields the same result regardless of the order of the two vectors (in other words, $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$), the cross product introduces a negative sign when the vectors are swapped ($\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$).

**Example 1.6:** What is the cross product of $\mathbf{A}=3\mathbf{a}_x+2\mathbf{a}_y+1\mathbf{a}_z$ and $\mathbf{B}=4\mathbf{a}_x+1\mathbf{a}_y+5\mathbf{a}_z$?

### 1.8 Summary

- Scalars have only a magnitude, while vectors have a magnitude and a direction.
- Unit vectors are represented by the letter $\mathbf{a}$ with a subscript. They have a magnitude of one and point in an important direction. In rectangular coordinates, any vector can be represented as a weighted sum of three unit vectors.
- The magnitude of a vector can be found using: $|\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$
- Adding or subtracting vectors requires that you add or subtract their components when in rectangular form.
- There are two ways to find the dot product: $\mathbf{A} \cdot \mathbf{B} = AB \cdot \cos \theta$ and $\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$
- You can combine these equations to find the angle between two vectors.
- The magnitude of the cross product is found using: $|\mathbf{A} \times \mathbf{B}| = AB \cdot \sin \theta$
- The direction of the cross product is found using the right-hand rule.
- Both the magnitude and the direction of the cross product can be found in rectangular coordinates using:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad \text{or} \quad \mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y) \mathbf{a}_x + (A_z B_x - A_x B_z) \mathbf{a}_y + (A_x B_y - A_y B_x) \mathbf{a}_z$$